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ON RIEMANNIAN SPACES WITH SPHERICAL SYMMETRY ABOUT A LINE, AND THE CONDITIONS FOR ISOTROPY IN GENERAL RELATIVITY

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[Received 23 August 1934]

Introduction. Riemannian spaces having spherical symmetry about a line are interesting geometrically and are important in that they can be applied to an important class of problems of general relativity. I propose, therefore, to consider such spaces, and in particular, to find under what conditions a space has local spherical symmetry about a given curve.

In dealing with physical structures known to have 'spatial' spherical symmetry, it is often assumed, for simplicity, that pressure is everywhere isotropic. This assumption imposes a certain restriction on the appropriate line element, but many writers impose more restrictions than are necessary, although, as we shall see, there is no apparent physical interpretation that can be given to the additional restriction. It may therefore be not without interest to find under what conditions a spherically symmetric 4-space represents a universe in which pressure is isotropic, and to compare these with the conditions assumed by other writers on the subject.

I shall assume a knowledge† of the tensor calculus and of the more elementary part of Riemannian geometry, and shall follow, as far as is possible, the notations already introduced by other writers.

1. Spherical symmetry about a line. A space V_n has *spherical symmetry about a curve* C, if any rotation about C transforms V_n into itself. Thus, if V_n is referred to coordinates specified by orthogonal $(n-1)$-uples orthogonal to C at points of C, the corresponding metric is unaltered in form when all the $(n-1)$-uples are given the same arbitrary rotation about C, i.e. undergo the same orthogonal transformation.

We first observe that, as there can be no preferential direction orthogonal to C, the curve must be a geodesic. A convenient system

† See L. P. Eisenhart, *Riemannian Geometry* (1926), and O. Veblen, *Cambridge Tract*, No. 24 (1927).

of coordinates can now be defined as follows. Let P_0 be a fixed point of C, and let $(h^i_\sigma)_0$ ($\sigma = 1, 2,\ldots, n-1$) be $n-1$ unit vectors at P_0, mutually orthogonal and orthogonal to C. Let τ be the arcual distance of a point P of C measured from P_0, and let h^i_σ ($\sigma = 1, 2,\ldots, n-1$) be the $n-1$ vectors at P derived from the vectors $(h^i_\sigma)_0$ by Levi-Civita parallel transport along C. Then the vectors h^i_σ are mutually orthogonal and orthogonal to C at P, and the components of these vectors are functions of τ. In the space $V_{n-1}(\tau)$ defined by the geodesics of V_n orthogonal to C at the point $P(\tau)$, let z^σ ($\sigma = 1, 2,\ldots, n-1$) be the system of normal coordinates, with origin at P and parametric directions h^i_σ at P. Then we can refer V_n to coordinates τ, z^σ, this system being completely determined by the geodesic C and the initial vectors $(h^i_\sigma)_0$ at P_0.

A rotation about C is now given by an orthogonal transformation of the z's, i.e.

$$z^\sigma = \xi^\sigma_{\sigma'} z'^{\sigma'} \qquad (\sigma = 1,2,\ldots,n-1), \tag{1}$$

where the ξ's are constants satisfying equations of the form

$$\delta_{\rho\sigma}\,\xi^\rho_{\rho'}\,\xi^\sigma_{\sigma'} = \delta_{\rho'\sigma'}, \tag{2}$$

and $\delta_{\rho\sigma}$ is defined by

$$\delta_{\rho\rho} = 1, \quad \delta_{\rho\sigma} = 0 \qquad (\rho, \sigma = 1,2,\ldots,n-1;\ \rho \neq \sigma). \tag{3}$$

Hence, for spherical symmetry about C, the metric of V_n must be unaltered in form by all such transformations. The metric can be written

$$ds^2 = g'_{00}\,d\tau^2 + 2g'_{0\sigma}\,d\tau dz^\sigma + g'_{\rho\sigma}\,dz^\rho dz^\sigma, \tag{4}$$

and for spherical symmetry, we must therefore have

$$g'_{00}(\xi z') = g'_{00}(z'),$$
$$g'_{0\sigma}(\xi z')\xi^\sigma_{\sigma'} = g'_{0\sigma'}(z'),$$
$$g'_{\rho\sigma}(\xi z')\xi^\rho_{\rho'}\,\xi^\sigma_{\sigma'} = g'_{\rho'\sigma'}(z') \qquad (\rho', \sigma' = 1,2,\ldots,n-1)$$

for all ξ's satisfying (2). It can easily be verified that the components g'_{ij} must be of the form

$$g'_{00} = a,$$
$$g'_{0\sigma} = bz_\sigma, \qquad z_\sigma = \delta_{\rho\sigma}z^\rho = z^\sigma,$$
$$g'_{\rho\sigma} = cz_\rho z_\sigma + e\delta_{\rho\sigma},$$

where a, b, c, e are functions of τ and ω, and $\omega = \delta_{\rho\sigma}z^\rho z^\sigma$. The functions a, b, c, e are not all arbitrary, for the sub-spaces $\tau =$ constant

must contain the geodesics orthogonal to C, the z's being normal coordinates in these sub-spaces. Hence we find

$$b = 0, \qquad c\omega + e = 1.$$

Also, the curve $z^\sigma = 0$ is the geodesic C, of arc τ, so that $(a)_{\omega=0} = 1$. We have, therefore,

$$\begin{aligned} g'_{00} &= a, \qquad (a)_{\omega=0} = 1, \\ g'_{0\sigma} &= 0, \\ g'_{\rho\sigma} &= \delta_{\rho\sigma} + c(\delta_{\rho\mu}\delta_{\sigma\nu} - \delta_{\rho\sigma}\delta_{\mu\nu})z^\mu z^\nu. \end{aligned} \tag{5}$$

Writing $\omega = r^2$, then

$$\delta_{\rho\mu} z^\mu dz^\rho = r\,dr, \qquad \delta_{\rho\sigma} dz^\rho dz^\sigma = dr^2 + r^2 d\varsigma^2, \tag{6}$$

where $d\varsigma^2$ is the metric of the unit hypersphere in flat space E_{n-2}. Substituting (5) and (6) in (4), the metric can be written

$$ds^2 = a\,d\tau^2 + dr^2 + er^2\,d\varsigma^2, \tag{7}$$

where a, e are functions of τ and r, and $(a)_{r=0} = 1$.

2. Conditions for spherical symmetry about a line. To find in tensor form the conditions to be satisfied by V_n in order that the space shall have spherical symmetry about a given geodesic C, we require the transformation from a general system of coordinates X^i to the system τ, z^σ. The transformation from (X) to normal coordinates (y) with origin at the point (X_0) is

$$X^i = X^i_0 + y^i - \frac{1}{2!}(\Gamma^i_{jk})_0 y^j y^k - \frac{1}{3!}(\Gamma^i_{jkl})_0 y^j y^k y^l - \dots, \tag{8}$$

where the coefficients are certain well-known functions of the Christoffel symbols Γ^i_{jk} and their derivatives, and are evaluated at (X_0). Hence, if the geodesic C, of arc τ, is given by the equations

$$X^i = x^i(\tau), \tag{9}$$

the transformation from (X) to (τ, z) is

$$X^i = f^i(x,y), \qquad x^i = x^i(\tau), \qquad y^i = h^i_\sigma z^\sigma, \tag{10}$$

where

$$f^i(x,y) = x^i + y^i - \frac{1}{2!}\Gamma^i_{jk} y^j y^k - \dots, \tag{11}$$

and x^α is substituted for X^α in the expressions $\Gamma^i_{jkl\dots}$. In (10) the h's are functions of τ, and, if we write $h^i = dx^i/d\tau$ so that h^i is the unit vector tangent to C, the components h^i_σ, h^i satisfy the equations

$$dh^i_\sigma + \Gamma^i_{jk} h^j_\sigma\,d\tau = 0, \tag{12}$$

$$g_{ij} h^i_\rho h^j_\sigma = \delta_{\rho\sigma}, \quad g_{ij} h^i_\sigma h^j = 0, \quad g_{ij} h^i h^j = 1 \quad (\rho, \sigma = 1, 2, \dots, n-1), \tag{13}$$

where g_{ij} is the fundamental tensor at (x) for the system (X). Here we have assumed the signature of V_n to be n, but the procedure is similar when the signature is $\pm(n-2)$.

Writing $f_i^\alpha = \partial f^\alpha/\partial y^i$, we have, from (10),

$$dX^\alpha = \frac{\partial f^\alpha}{\partial x^k}\,dx^k + f_i^\alpha(dh_\sigma^i z^\sigma + h_\sigma^i\,dz^\sigma),$$

i.e., by (12),

$$dX^\alpha = \left(\frac{\partial f^\alpha}{\partial x^k} - f_i^\alpha\,\Gamma_{jk}^i\,h_\sigma^j z^\sigma\right)h^k\,d\tau + f_i^\alpha\,h_\sigma^i\,dz^\sigma.$$

Hence we can write

$$dX^\alpha = f_i^\alpha(V_k^i\,h^k\,d\tau + h_\sigma^i\,dz^\sigma),$$

where
$$V_k^i = \bar{f}_j^i\,\frac{\partial f^j}{\partial x^k} - \Gamma_{jk}^i\,y^j, \tag{14}$$

and $\bar{f}_j^i$ is the normalized co-factor of f_i^j in (f_β^α). The metric of V_n is therefore

$$ds^2 = \bar{g}_{\alpha\beta}\,dX^\alpha dX^\beta = \bar{g}_{\alpha\beta}f_i^\alpha f_j^\beta(V_k^i\,h^k\,d\tau + h_\rho^i\,dz^\rho)(V_l^j\,h^l\,d\tau + h_\sigma^j\,dz^\sigma), \tag{15}$$

where $\bar{g}_{\alpha\beta}$ is the fundamental tensor for the system (X). Now from (10) we see that the fundamental tensor g_{ij}^* for the normal system (y), with origin at (x), is given by

$$g_{ij}^* = \bar{g}_{\alpha\beta}f_i^\alpha f_j^\beta.$$

Substituting in (15) and comparing the form (15) with the form (4), we find that the components of the fundamental tensor for the system (τ, z) are given by

$$g'_{00} = g_{ij}^*\,V_k^i\,V_l^j\,h^k h^l, \qquad g'_{0\sigma} = g_{ij}^*\,V_k^i\,h^k h_\sigma^j, \qquad g'_{\rho\sigma} = g_{ij}^*\,h_\rho^i\,h_\sigma^j, \tag{16}$$

where x, y are expressed in terms of τ, z by (10).

The expression V_k^i defined by (14) is interesting in tensor theory, for it can be shown that, if V_k^i is expanded by means of (11) as a series in (y), then the coefficients, as functions of (x), are components of tensors. Hence the expression generates a sequence of tensors constructed from the fundamental tensor. It can easily be verified that the first few terms are

$$V_k^i = \delta_k^i + \tfrac{1}{3}R_{\alpha\beta k}^i\,y^\alpha y^\beta + \tfrac{1}{12}R_{\alpha\beta k.\gamma}^i\,y^\alpha y^\beta y^\gamma + \ldots. \tag{17}$$

We have now found, in general form, the components g'_{ij}. The conditions for spherical symmetry about C are therefore that these components may be expressed in the form (5). These conditions are found by expanding the expressions (16) in terms of (z) by means

of (17) and (10)† and equating the coefficients to the corresponding coefficients in (5), where a and c are expanded as series in ω. The arbitrary functions of τ arising from the expansions of a and c are then eliminated, and there remains a sequence of tensor equations to be satisfied at all points of C. These conditions soon become complicated, and we shall consider in detail only those given by the second-order terms. If these are satisfied, we shall say that the space has *local spherical symmetry* about C. Thus, for local symmetry, we are concerned only with the space in the neighbourhood of C which is such that cubed distances from C may be neglected.

Writing $\qquad a = 1+a_0\omega+\ldots, \qquad (c)_{\omega=0} = c_0,$

it can be shown by the above method that, for local symmetry, the components of the curvature tensor at points of C must satisfy the equations

$$R_{hijk} = (a_0+3c_0)(g_{hk}h_i h_j+g_{ij}h_h h_k-g_{hj}h_i h_k-g_{ik}h_h h_j)-$$
$$-3c_0(h_{hk}g_{ij}-h_{hj}g_{ik}). \tag{18}$$

Contracting for h and k, the Ricci tensor $R_{ij} = g^{hk}R_{hijk}$ is given by

$$R_{ij} = \alpha h_i h_j+\beta g_{ij}, \tag{19}$$

where $\qquad \alpha = (n-2)(a_0+3c_0), \qquad \beta = a_0-3(n-2)c_0, \qquad (20)$

and contracting again, $\qquad R = \alpha+n\beta. \qquad (21)$

Substituting in (18), we get

$$R_{hijk} = \frac{1}{n-2}(g_{hk}R_{ij}+g_{ij}R_{hk}-g_{hj}R_{ik}-g_{ik}R_{hj})-$$
$$-\frac{R}{(n-1)(n-2)}(g_{hk}g_{ij}-g_{hj}g_{ik}) \tag{22}$$

i.e. $\qquad C_{hijk} = 0 \qquad (h,i,j,k = 0,1,\ldots,n-1), \qquad (23)$

where C_{hijk} are the components of the conformal tensor.‡ Equations (22) and (19) reproduce equations (18). Hence, *for local spherical symmetry about a geodesic C, the components of the conformal tensor must vanish at all points of C, and the Ricci tensor must be expressible in the form* (19), *where α, β are functions of τ and h^i is the unit tangent vector of C*; these conditions are necessary and sufficient.

† The expansion of g^*_{ij} is $g_{ij}+\sum_{s=2}^{\infty}\frac{1}{s!}A_{ijk_1\ldots k_s}y^{k_1}\ldots y^{k_s}$, where $A_{ijk_1\ldots k_s}$ are the components of the sth normal tensor, evaluated at (x). The normal tensors can be expressed in terms of the curvature tensor, and, in particular, $3A_{ijhk} = R_{hijk}+R_{hjik}$.

‡ Eisenhart, op. cit., § 28.

3. Consider now a system of geodesics such that a member of the system passes through each point of V_n. We shall prove that, if the space has spherical symmetry about each of these geodesics, coordinates can be chosen so that the metric of V_n has the form

$$ds^2 = dt^2 + U(t)\, d\varsigma^2, \tag{24}$$

where $d\varsigma^2$ is the metric of a space C_{n-1} of constant curvature, independent of t. The given geodesics are then the orthogonal trajectories of the surfaces $t =$ constant. We shall, in fact, prove a more general statement, which is that the space must be of the above form, if it has local spherical symmetry about each geodesic.

From the results of § 2 we see that, as the components of the conformal tensor must vanish at all points of each geodesic, they must vanish at each point of V_n, i.e. V_n must be conformal to a flat space.† Further, the Ricci tensor must be of the form (19), where h^i is now a vector field satisfying

$$g_{ij}h^i h^j = 1, \quad h^i{}_{,k}h^k = 0 \qquad (i = 0, 1,..., n-1), \tag{25}$$

and α, β are functions of position. Writing

$$F_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}, \tag{26}$$

equations (22) become

$$R_{hijk} = \frac{1}{n-2}(g_{hk}F_{ij} + g_{ij}F_{hk} - g_{hj}F_{ik} - g_{ik}F_{hj}), \tag{27}$$

and we also have‡

$$F_{ij,k} = F_{ik,j}. \tag{28}$$

Equations (19) can now be written in the form

$$F_{ij} = Ah_i h_j + Bg_{ij}, \tag{29}$$

where A, B are functions of position, and substituting in (28) we find,§ after contracting and substituting from (25),

$$\frac{\partial A}{\partial x_i} = \frac{\partial A}{\partial x^k} h^k h_i, \qquad \frac{\partial B}{\partial x^i} = \frac{\partial B}{\partial x^k} h^k h_i, \tag{30}$$

$$h_{i,j} = h_{j,i} = \frac{1}{A}\frac{\partial B}{\partial x^k} h^k (g_{ij} - h_i h_j) \qquad (i,j = 0, 1,..., n-1). \tag{31}$$

From equations (30), A and B cannot be independent functions, and the vectors h^i must be normal to the surfaces A (or B) $=$ constant,

† Eisenhart, op. cit., § 28.

‡ Eisenhart, op. cit., 92, (28.19).

§ If $A = 0$, V_n must be a space of constant curvature, and it is known that coordinates can be chosen so that the metric of such a space takes the form (24).

i.e. the given geodesics must be orthogonal to these surfaces. We can therefore choose coordinates t, x^σ ($\sigma = 1, 2,..., n-1$) so that the metric of V_n takes the form

$$ds^2 = dt^2 + g_{\rho\sigma} dx^\rho dx^\sigma, \tag{32}$$

and A, B are functions of t only. The given geodesics are now orthogonal to the surfaces $t =$ constant, whence, from (32), the components h^i, h_i are (1, 0,..., 0). Calculating $h_{i,j}$, equations (31) become

$$\frac{\partial}{\partial t}(g_{\rho\sigma}) = \frac{2}{A}\frac{dB}{dt} g_{\rho\sigma},$$

and these differential equations can be integrated to give

$$g_{\rho\sigma} = U f_{\rho\sigma} \qquad (\rho, \sigma = 1, 2,..., n-1), \tag{33}$$

where U is a function of t only and the f's are independent of t.

Substituting (29) and (33) in (27), we find

$$R_{\rho\mu\nu\sigma} = \frac{2}{n-2} B U^2 (f_{\rho\sigma} f_{\mu\nu} - f_{\rho\nu} f_{\mu\sigma}) \qquad (\rho, \mu, \nu, \sigma = 1, 2,..., n-1). \tag{34}$$

Calculating these components from (32) and (33),

$$R_{\rho\mu\nu\sigma} = U R'_{\rho\mu\nu\sigma} + \frac{1}{4}\left(\frac{dU}{dt}\right)^2 (f_{\rho\sigma} f_{\mu\nu} - f_{\rho\nu} f_{\mu\sigma}), \tag{35}$$

where $R'_{\rho\mu\nu\sigma}$ are the components of the curvature tensor for the $(n-1)$-space

$$ds^2 = f_{\rho\sigma} dx^\rho dx^\sigma. \tag{36}$$

Hence, from (34) and (35),

$$R'_{\rho\mu\nu\sigma} = K(f_{\rho\sigma} f_{\mu\nu} - f_{\rho\nu} f_{\mu\sigma}),$$

whence the space with the metric (36) is a space of constant curvature. Substituting (33) and (36) in (32), we get the form (24), which proves our statement, for it is evident that the space with the metric (24) has spherical symmetry about each of the geodesics orthogonal to the surfaces $t =$ constant.

4. Spherical symmetry in general relativity. Spherical symmetry about a line is important in a certain class of problems of general relativity, for, if a given physical structure has spherical symmetry about some particle, the corresponding 4-space must have spherical symmetry about this particle's world-line. Many problems† are concerned with a non-uniform distribution of matter spherically

† See, for example, G. C. McVittie, *Monthly Notices R.A.S.* 92 (1932), 500, and 93 (1933), 325.

symmetric about the 'spatial' origin (e.g. a massive particle at the origin), in which case the space-time given by the external solution has spherical symmetry, and may have singularities at or near the origin. We have not considered spaces with singularities, but it is at once seen that part of the analysis of § 1 is still valid.

Coordinates τ, z^σ $(\sigma = 1, 2, 3)$ are now defined to be such that a rotation about the spatial origin is given by (1) subject to (2), and we find, as in § 1, that the metric must be of the form

$$ds^2 = a\,d\tau^2 + 2b\,d\tau \sum z\,dz + c(\sum z\,dz)^2 + e \sum (dz)^2, \tag{37}$$

where a, b, c, e are arbitrary functions of τ and $\sum (z)^2$. Transforming from coordinates (z) to polar coordinates r', θ, ϕ, we have

$$\sum (z)^2 = r'^2, \qquad \sum z\,dz = r'dr',$$
$$\sum (dz)^2 = dr'^2 + r'^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

and the form (37) becomes

$$ds^2 = a\,d\tau^2 + 2b\,d\tau dr' + (e + cr'^2)\,dr'^2 + er'^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{38}$$

where a, b, c, e are functions of τ and r'. A singular region about the origin will be given by $r' \leqslant r'_0$ for some value of r'_0 possibly depending on τ, so that some or all of a, b, c, e will become infinite at $r' = r'_0$.

It has been shown† that there exists a transformation $\tau = \tau(t, r)$, $r' = r'(t, r)$ such that the metric (38) becomes

$$ds^2 = H(t, r)\,dt^2 - F(t, r)\{dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\}, \tag{39}$$

and this is the most convenient metric that can be assumed for general structures having spherical symmetry. Any restrictions on the forms of H and F must result from the physical conditions satisfied by the particular structure under consideration.

5. Isotropic systems. The field equations connecting the energy tensor T^i_j and the fundamental tensor are‡

$$\kappa T^i_j = -R^i_j + (\tfrac{1}{2}R - \lambda)\delta^i_j, \qquad R^i_j = g^{ik}R_{jk}, \tag{40}$$

where κ is Einstein's constant of gravitation and λ the cosmical constant. If the physical system is isotropic (i.e. the matter behaves as a perfect fluid), it has been shown§ that the energy tensor can be written

$$T^i_j = (\rho_{00} + p)g_{jk}v^i v^k - p\delta^i_j, \tag{41}$$

† R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (1934), 240. It must be remembered that we are now concerned with a 4-space of signature -2.
‡ Tolman, op. cit., 189.
§ Tolman, op. cit., 217.

where ρ_{00} and p are the proper macroscopic density and pressure, and the quantities v^i are the components of a unit contravariant vector field, defining the 4-dimensional lines of flow (the fundamental world-lines). Allowing ρ_{00}, p, v^i to be arbitrary, equations (41) impose certain restrictions on the form of the energy tensor and hence, by (40), on the fundamental tensor, and these are therefore the conditions for isotropic pressure. If they are satisfied, the quantities which determine the physical state can be found from (41) and (40) and are, in general, unique.

Let us now find the conditions to be satisfied by space-time with spherical symmetry in order that the corresponding physical system shall be everywhere isotropic. Choosing coordinates t, r, θ, ϕ as in § 4, then, from (39),

$$g_{00} = H, \qquad g_{11} = -F, \qquad g_{22} = -Fr^2, \qquad g_{33} = -Fr^2\sin^2\theta,$$
$$g_{ij} = 0 \quad (i \neq j). \tag{42}$$

The components T^i_j can be readily calculated† from (42), and we find identically

$$T^2_2 = T^3_3,$$
$$T^0_2 = T^2_0 = T^0_3 = T^3_0 = 0, \tag{43}$$
$$T^1_2 = T^2_1 = T^1_3 = T^3_1 = T^2_3 = T^3_2 = 0.$$

Substituting (42) and (43) in (41), we find $v^2 = v^3 = 0$, and the remaining independent equations are

$$T^0_0 = (\rho_{00}+p)H(v^0)^2-p, \qquad T^1_1 = -(\rho_{00}+p)F(v^1)^2-p,$$
$$T^2_2 = -p, \qquad T^0_1 = -(\rho_{00}+p)Fv^0v^1. \tag{44}$$

We have here assumed that $\rho_{00}+p$ does not vanish, thus excluding the empty universe of de Sitter. The quantities ρ_{00}, p, v^0, v^1 can at once be eliminated from (44), and there remains the equation

$$H(T^0_1)^2+F(T^2_2-T^0_0)(T^2_2-T^1_1) = 0 \tag{45}$$

to be satisfied by H and F and their derivatives. *This, then, is the necessary and sufficient condition that the metric* (39) *shall correspond to an isotropic system.*

To find ρ_{00}, p, v^0, v^1, we remember that v^i is a unit time-like vector, whence from (42),

$$H(v^0)^2-F(v^1)^2 = 1. \tag{46}$$

† H. Dingle, *Proc. National Acad. Sci.* 19 (1933), 559.

Hence, from (44) and (46),

$$\rho_{00} = T_0^0 + T_1^1 - T_2^2, \qquad p = -T_2^2, \tag{47}$$

$$v^0 = \left\{\frac{T_0^0 - T_2^2}{H(T_0^0 + T_1^1 - 2T_2^2)}\right\}^{\frac{1}{2}}, \qquad v^1 = \pm\left\{\frac{T_2^2 - T_1^1}{F(T_0^0 + T_1^1 - 2T_2^2)}\right\}^{\frac{1}{2}}, \tag{48}$$

where the sign of v^1 is that of $-T_1^0$.

6. We have shown that the most general form of space-time with spherical symmetry depends upon two arbitrary functions H and F, and for the physical system to be isotropic it is necessary and sufficient that these functions shall satisfy the one condition given by (45). This general result has apparently been missed by all writers on the subject, for it is usually assumed† that, in space-time with the metric (39), the lines of flow are the curves orthogonal to the surfaces t = constant, i.e. that $v^1 = v^2 = v^3 = 0$, in which case the conditions for isotropic pressure are

$$T_1^1 = T_2^2 \ (= T_3^3), \qquad T_1^0 = 0. \tag{49}$$

If these conditions are satisfied, the more general condition (45) is satisfied, but a further restriction has been imposed on the metric (39) and hence on the physical structure. There appears, however, to be no reason for this restriction, except that it simplifies the analysis.

In order to excuse the above simplification, it is sometimes argued that the vanishing of the quantities v^1, v^2, v^3 ensures the absence of mass-motion. This, however, is very arbitrary, for it implies that the motion should be measured by the observers with world-lines orthogonal to the surfaces t = constant. The argument would, perhaps, carry weight if these observers were uniquely defined by the geometry and hence by the physical system, but it is at once seen that this is not the case. For it is possible to transform from coordinates t, r to t', r' so that the metric (39) transforms into

$$ds^2 = H'\,dt'^2 - F'\{dr'^2 + r'^2(d\theta^2 + \sin^2\theta\,d\phi^2)\},$$

where H', F' are some functions of t' and r'. Thus, under such a transformation, the metric is unchanged in *form*, and hence we cannot distinguish between the surfaces t = constant and the surfaces t' = constant. In the above transformation t' is not necessarily dependent only on t, so that the two systems of surfaces are not necessarily coincident. Hence, the observers mentioned above are not defined uniquely by the geometry.

† See, for example, H. Dingle, *Monthly Notices R.A.S.* 94 (1934), 134.

H. Dingle has recently made a detailed analysis of the systems corresponding to the metric (39) subject to the conditions (49). It would be of interest and value if a similar analysis were made of the more general case where (45) replaces (49).

7. We have remarked that, when a coordinate system is chosen so that the metric is of the form (39), the general condition for isotropic pressure is (45) and not (49). This may be demonstrated by finding the coordinate system and corresponding metric for which the conditions for isotropic pressure take the form (49) and showing that such a metric cannot in general be of the form (39). It is possible that in some applications this particular coordinate system will be more convenient than the systems already mentioned.

If the new system is t^*, r^*, θ, ϕ, it is required that these coordinates shall be orthogonal and that the lines of flow shall be orthogonal to the surfaces $t^* = \text{constant}$. The transformation from the system t, r, θ, ϕ to the new system is of the form

$$t^* = f(t,r), \qquad r^* = g(t,r), \tag{50}$$

and must be such that the metric (39) is transformed into

$$ds^2 = H^*(dt^*)^2 - F^*(dr^*)^2 - G^*(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{51}$$

where H^*, F^*, G^* are some functions of t^* and r^*. Also, $f(t,r)$ must be such that the vectors v^i are orthogonal to the surfaces $f = \text{constant}$, whence

$$Hv^0 \frac{\partial f}{\partial r} = -Fv^1 \frac{\partial f}{\partial t}, \tag{52}$$

i.e. $f = \text{constant}$ is an integral of the differential equation

$$Fv^1\, dr - Hv^0\, dt = 0. \tag{53}$$

Substituting for t and r from (50) in (49) and equating to zero the coefficient of $dt^* dr^*$, we find

$$H \frac{\partial f}{\partial r} \frac{\partial g}{\partial r} = F \frac{\partial f}{\partial t} \frac{\partial g}{\partial t}, \tag{54}$$

whence, from (52), $g = \text{constant}$ is an integral of the differential equation

$$v^0\, dr - v^1\, dt = 0. \tag{55}$$

The functions f and g giving the required transformation are therefore determined as integrals of equations (53) and (55), and we find that the coefficients H^*, F^*, G^* are given by

$$\begin{aligned} H^* &= H^2(v^0)^2(\partial f/\partial t)^{-2}, \\ F^* &= HF(v^1)^2(\partial g/\partial t)^{-2}, \\ G^* &= Fr^2, \end{aligned} \tag{56}$$

where t, r are expressed in terms of t^* and r^*. We see therefore that, in general space-time with spherical symmetry, we can choose coordinates t^*, r^*, θ, ϕ so that the lines of flow are orthogonal to the surfaces $t^* =$ constant, and the metric takes the form (51), where H^*, F^*, G^* are functions of t^* and r^*. The conditions for isotropic pressure are now (49), and these give two equations relating H^*, F^*, and G^*.

Comparing (51) and (39), we now see that the usual assumption is that t^* and r^*, defined as above, can be chosen to satisfy

$$G^* = F^*(r^*)^2.$$

From (56) and (50), this equation is equivalent, in terms of t and r, to

$$\frac{1}{g}\frac{\partial g}{\partial t} = \pm H^{\frac{1}{2}}\frac{v^1}{r}, \tag{57}$$

whence, from (54) and (52),

$$\frac{1}{g}\frac{\partial g}{\partial r} = \mp H^{\frac{1}{2}}\frac{v^0}{r}. \tag{58}$$

But equations (57) and (58) are compatible only if

$$\frac{\partial}{\partial t}\left(H^{\frac{1}{2}}\frac{v^0}{r}\right)+\frac{\partial}{\partial r}\left(H^{\frac{1}{2}}\frac{v^1}{r}\right) = 0, \tag{59}$$

and substituting for v^0, v^1 in terms of H and F from (48), we see that equation (59) is a further restriction imposed on H and F in the general metric (39). Hence, in general, coordinates cannot be chosen so that the metric is of the form (39), the conditions for isotropic pressure being (49).

It would be of interest to find some physical interpretation of equation (59), but such an interpretation is not apparent. It appears as if the nature of the usual restriction is purely analytical.

8. Local spherical symmetry. It is generally understood that when spherically symmetric space-time is assumed to represent some feature of the universe, it does not in general give an accurate representation of the whole universe, for it is assumed that the remainder of the universe is smoothed out. Such forms as have been discussed above are therefore required to correspond to the physical system only in the neighbourhood of the origin. Hence, if it is assumed that there is no singularity at or near the origin, the conditions under which such a spherically symmetric region can exist may be found by the methods of § 2.

If a region has local spherical symmetry (as defined in § 2), the corresponding metric can be used to find the physical state only at the origin, third-order terms having been neglected. We have shown that, if such a region exists, the components of the conformal tensor must vanish at the origin and, at the origin, the Ricci tensor must be expressible in the form (19). Writing v^i for h^i, we see from (40) that, if the second condition is satisfied, the energy tensor can be written in the form (41) at the origin. Hence, pressure is isotropic at the origin, and this point describes a line of flow. We see therefore that, *if the pressure is isotropic and the components of the conformal tensor vanish at all points of a line of flow, the space-time has local spherical symmetry about this curve.* The conditions for the existence of a more extended region of symmetry could similarly be found by the methods of § 2.

In space-time the form (24) is replaced by

$$ds^2 = dt^2 - \{R(t)\}^2 d\varsigma^2, \tag{60}$$

where $d\varsigma^2$ is the metric of a 3-space of constant curvature. This form is at once recognized as the metric of the homogeneous isotropic universe.† Hence, from § 3, *the only universe in which there is local spherical symmetry about each fundamental observer is the homogeneous isotropic universe.*

† H. P. Robertson, *Reviews of Modern Physics*, 5 (1932), 65.

ON ANALYTIC FUNCTIONS REGULAR IN THE UNIT CIRCLE (II)

By MARY L. CARTWRIGHT (*Cambridge*)

[Received 5 September 1934]

1.1. LET $$f(z) = f(re^{i\theta}) = u(r,\theta)+iv(r,\theta)$$
be regular for $|z| < 1$, and let $f(0) = 0$. We write

$$M(r) = \max_{0\leqslant\theta<2\pi} |f(re^{i\theta})|, \quad M_p(r,u) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |u(r,\theta)|^p\, d\theta\right\}^{1/p} \quad (p > 0),$$

and $\quad u_+(r,\theta) = u(r,\theta) \quad (u > 0), \qquad u_+(r,\theta) = 0$ otherwise.

In an earlier paper* I showed that, if

$$u_+(r,\theta) < A(1-r)^{-\alpha} \qquad (r < 1;\ \alpha > 0),$$

then, for $r < 1$,

$$M(r) < K(\alpha)A(1-r)^{-1} \qquad (\alpha < 1), \tag{1.11}$$

$$M(r) < KA(1-r)^{-1}\left(\log\frac{1}{1-r}\right)^2 \qquad (\alpha = 1), \tag{1.12}$$

$$M(r) < K(\alpha)A(1-r)^{-\alpha} \qquad (\alpha > 1). \tag{1.13}$$

The chief object of this paper is to prove a similar result for $M_p(r,u_+)$ and $M_p(r,f)$, where $p \geqslant 1$.

It should be observed that, if $p = 1$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\, d\theta = u(0) = 0; \tag{1.14}$$

and so $M_1(r,u_+)$ and $M_1(r,u)$ are of the same order. Hence the results for $M_1(r,u_+)$ and $M_1(r,f)$ can be deduced from the results of Hardy and Littlewood† for conjugate functions. I shall begin by proving various theorems which can be deduced more or less trivially from (1.14) by the methods used for conjugate functions. If $p > 1$, these are not the best possible results; but, by extending the methods used to obtain (1.12) and (1.13), we can prove the following theorem. The method used for (1.11) seems to be inapplicable to mean values.

* M. L. Cartwright, *Quart. J. of Math.* (Oxford), 4 (1933), 246–57. I shall refer to this paper as I.

† G. H. Hardy and J. E. Littlewood, *J. für Math.* 167 (1931), 405–23.

THEOREM I. *If* $$M_p(r, u_+) < A(1-r)^{-\alpha}, \tag{1.15}$$
where $p \geqslant 1$, $\alpha > 1$, *for* $r < 1$, *then*
$$M_p(r,f) < K(\alpha, p)A(1-r)^{-\alpha} \tag{1.16}$$
for $r < 1$.

I hope to give the corresponding results for $\alpha \leqslant 1$ later.

2.1. The constants* K, $K(\alpha)$, $K(\alpha, p)$ in (1.11), (1.12) and throughout the paper are independent of the function. Mr. Ingham pointed out to me that in Lemmas 1 and 2 of I the constants $K(\delta)$ depend on the function $F(z)$. As we shall use (1.12) and (1.13) which were obtained from those lemmas, we must begin by removing the imperfection. In Lemma 1 of I it is enough to suppose that $|F(re^{i\theta})| > 1$ for $|\theta| \leqslant \beta$, $1 \leqslant r \leqslant 2l$; we can then show that $K(\delta)$ is independent of the function, and the altered form of the lemma is sufficient for the application. However, it seems much simpler to replace the lemma by the following lemma. The $F(z)$ in this lemma may be compared with the logarithm of the $F(z)$ in Lemma 1 of I.

LEMMA 1. *Suppose that* $F(z) = U(r,\theta)+i\,V(r,\theta)$ *is regular, and that*
$$U(r,\theta) < B_1 r^{\alpha}, \tag{2.11}$$
for $r \geqslant l$, $|\theta| \leqslant \beta$, *where* $\beta > \frac{1}{2}\pi/\alpha$. *Suppose also that*
$$|F(le^{i\theta})| < B_2 \tag{2.12}$$
for $|\theta| \leqslant \beta$. *Then, given any positive* δ, *we can choose* $K(l,\delta)$, $K(\alpha,\beta,\delta)$ *so that*
$$U(r,\theta) > -K(\alpha,\beta,\delta)B_1 r^{\alpha} - K(l,\delta)B_2 r^{\pi/2\beta}$$
for $|\theta| \leqslant \beta-\delta$, $r \geqslant l$.

We use Carleman's formula† in a modified form.

CARLEMAN'S FORMULA. *Suppose that* $F(z) = U(r,\theta)+i\,V(r,\theta)$ *is regular for* $l \leqslant r \leqslant R$, $|\theta| \leqslant \beta$. *Then*
$$\frac{l}{2\pi}\int_{-\beta}^{\beta}\Big\{U(l,\theta)(l^{-\pi/2\beta-1}+R^{-\pi/2\beta-1})\cos\frac{\pi\theta}{2\beta}-$$
$$-V(l,\theta)(l^{-\pi/2\beta-1}-R^{-\pi/2\beta-1})\sin\frac{\pi\theta}{2\beta}\Big\}\,d\theta$$

* The constants, K, $K(\alpha)$, $K(\alpha, p)$,..., need not be the same in each place, but A, B_1, B_2, C, H remain the same throughout a theorem and its proof.

† Cf. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, 1 (1925), 120, No. 178. The same method gives the form used here.

$$= \frac{1}{\pi R^{\pi/2\beta}} \int_{-\beta}^{\beta} U(R,\theta)\cos\frac{\theta\pi}{2\beta}\,d\theta +$$

$$+\frac{1}{2\pi}\int_{l}^{R} \{U(r,-\beta)+U(r,\beta)\}(r^{-\pi/2\beta}-r^{\pi/2\beta}R^{-\pi/\beta})\frac{dr}{r}.$$

Proof of Lemma 1. Using (2.11), (2.12) and Carleman's formula, we have

$$\frac{1}{\pi R^{\pi/2\beta}} \int_{-\beta}^{\beta} U(R,\theta)\cos\frac{\pi\theta}{2\beta}\,d\theta > -K(l)B_2 - \frac{B_1}{\pi}\int_{l}^{R} r^{\alpha-\pi/2\beta-1}\,dr$$

$$> -K(l)B_2 - B_1 K(\alpha,\beta)R^{\alpha-\pi/2\beta},$$

i.e. $$\int_{-\beta}^{\beta} U(R,\theta)\cos\frac{\pi\theta}{2\beta}\,d\theta > -K(l)B_2 R^{\pi/2\beta} - B_1 K(\alpha,\beta)R^{\alpha}.$$

It follows from this* and (2.11) that, given any positive δ, we can choose $K(l,\delta)$, $K(\alpha,\beta,\delta)$ so that

$$U(R,\theta) > -K(l,\delta)B_2 R^{\pi/2\beta} - K(\alpha,\beta,\delta)B_1 R^{\alpha} \tag{2.13}$$

for $R > l$ and $|\theta| \leqslant \beta$, except perhaps in a set of θ of measure less than $\frac{1}{2}\delta$. The proof may be completed by applying Carathéodory's theorem in a circle at whose centre (2.13) holds, just as in Lemma 2 of I.

Proof of (1.13). Put $1-z = 1/w$, and apply Lemma 1 to $F(w) = f(1-1/w)$. It follows from (3.11) of I that

$$|f(re^{i\theta})| \leqslant 4A(1-r)^{-\alpha-1}+A;$$

and so, if $\alpha > 1$, $F(z)$ satisfies the hypotheses of Lemma 1 for some β such that $\frac{1}{2}\pi/\alpha < \beta < \frac{1}{2}\pi$ with $l = \sec\beta$, $B_1 = K(\beta)A$, $B_2 = K(\beta)A$. Hence we have the required result as in I, but with constants which are independent of the function.

The same method also gives (1.12), and Lemma 2 of I may be put right by an additional hypothesis on $F(z)$ corresponding to (2.12).

3.1. We shall use the following results which are due to Hardy and Littlewood.†

* See I.

† For Lemma 2 see G. H. Hardy and J. E. Littlewood, *Math. Zeitschrift*, 28 (1928), 623, and 34 (1931), 406. For Lemma 3 see *J. für Math.* 167 (1931), 413. If $p > 1$, Lemma 3 holds with $\alpha = 0$, and is then derived from Riesz's theorem on conjugate functions.

LEMMA 2. *If* $p > 0$, $\alpha \geqslant 0$, *and*

$$M_p(r,f) < A(1-r)^{-\alpha} \qquad (r < 1),$$

then
$$M_q(r,f) < K(\alpha,p,q)A(1-r)^{-\alpha-1/p+1/q},$$

where $q > p$, *for* $r < 1$. *In particular, if* $q = \infty$, *we have*

$$M(r) < K(\alpha,p)A(1-r)^{-\alpha-1/p}.$$

LEMMA 3. *If* $p > 0$, $\alpha > 0$, *and*

$$M_p(r,u) < A(1-r)^{-\alpha} \qquad (r < 1),$$

then
$$M_p(r,f) < K(\alpha,p)A(1-r)^{-\alpha}$$

for $r < 1$.

3.2. We can now prove

THEOREM II. *If* $p \geqslant 1$, $\alpha > 0$, *and*

$$M_p(r,u_+) < A(1-r)^{-\alpha} \qquad (r < 1), \tag{3.21}$$

then, if $q \geqslant 1$, *we have*

$$M_q(r,f) < K(\alpha,p,q)A(1-r)^{-\alpha-1+1/q}, \quad \text{if} \quad \frac{1}{p}+\frac{1}{q} \geqslant 1, \tag{3.22}$$

$$M_q(r,f) < K(\alpha,p,q)A(1-r)^{-\alpha-1/p}, \quad \text{if} \quad \frac{1}{p}+\frac{1}{q} < 1,\ \alpha+\frac{1}{p} > 1, \tag{3.23}$$

$$M_q(r,f) < K(\alpha,p,q)A(1-r)^{-1}\{\log(1-r)\}^2,$$
$$\text{if} \quad \frac{1}{p}+\frac{1}{q} < 1,\ \alpha+\frac{1}{p} = 1, \tag{3.24}$$

$$M_q(r,f) < K(\alpha,p,q)A(1-r)^{-1}, \quad \text{if} \quad \frac{1}{p}+\frac{1}{q} < 1,\ \alpha+\frac{1}{p} < 1. \tag{3.25}$$

The inequality (3.22) holds whenever $q \geqslant 1$, but the others are better when $1/p+1/q < 1$. If $p = 1$, $q = 1$ or if $q = \infty$, the appropriate inequalities are the best possible with the exception of the inequality for $q = \infty$ given by (3.24). For $\alpha > 1$, $1 < p < \infty$, $1 < q < \infty$ the results are less good than those obtainable from (1.16), and it is probable that they can be improved for $0 < \alpha \leqslant 1$, $1 < p < \infty$, $1 < q < \infty$. A consideration of functions such as $-(1-z)^{-1}$, $-(1-z)^{-1}\{\log(1-z)\}^2$ shows, however, that we may have

$$M_p(r,f) > K(\alpha,p)A(1-r)^{-1+1/p}$$

when (3.21) holds for some $\alpha < -1+1/p$, and

$$M_p(r,f) > K(p)A(1-r)^{-1+1/p}\{\log(1-r)\}^2$$

when $\alpha = -1+1/p$.

Proof. It follows from Hölder's inequality that, if $p' = (p-1)/p$,

$$M_1(r, u_+) \leqslant M_p(r, u_+)\left\{\frac{1}{2\pi}\int\limits_0^{2\pi} d\theta\right\}^{1/p'} < A(1-r)^{-\alpha};$$

and so, by (1.14), we have

$$M_1(r, u) < A(1-r)^{-\alpha}.$$

Lemma 3 gives the required result for $q = 1$, and Lemma 2 completes (3.22).

For the other inequalities we use Poisson's formula,

$$u(r,\theta) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{u(R,\phi)(R^2-r^2)}{R^2+r^2-2Rr\cos(\phi-\theta)}\,d\phi$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi} u(R,\phi)P(R,r,\phi-\theta)\,d\phi,$$

where $r < R < 1$. Since $P(R,r,\phi-\theta) \geqslant 0$ for all ϕ and θ and all $r < R$, we have

$$u(r,\theta) \leqslant \frac{1}{2\pi}\int\limits_0^{2\pi} u_+(R,\phi)P(R,r,\phi-\theta)\,d\phi$$

$$\leqslant M_p(R, u_+)\left\{\frac{1}{2\pi}\int\limits_0^{2\pi} P^{p'}\,d\theta\right\}^{1/p'},$$

where $p' = (p-1)/p$, by Hölder's inequality. Hence, putting $R = \frac{1}{2}(1+r)$, we have

$$u(r,\theta) < KA(1-r)^{-\alpha+1}\left\{\frac{1}{\pi}\int\limits_0^{\pi} \frac{d\phi}{\{(1-r)^2+\phi^2\}^{p'}}\right\}^{1/p'}$$

$$< K(p)A(1-r)^{-\alpha-1+1/p'} = K(p)A(1-r)^{-\alpha-1/p}.$$

Using (1.11), (1.12), and (1.13), we have

$$M(r) < K(\alpha,p)A(1-r)^{-1}, \quad \text{if} \quad \alpha+1/p < 1,$$

$$M(r) < K(p)A(1-r)^{-1}\{\log(1-r)\}^2, \quad \text{if} \quad \alpha+1/p = 1,$$

$$M(r) < K(\alpha,p)A(1-r)^{-\alpha-1/p}, \quad \text{if} \quad \alpha+1/p > 1.$$

The inequalities (3.23), (3.24), and (3.25) follow from these by Hölder's inequality.

3.3. We can also get some comparatively trivial results for $0 < p < 1$.

THEOREM III. *If* $0 < p \leqslant 1$, $\alpha > 0$, *and*

$$M_p(r, u_+) < A(1-r)^{-\alpha} \qquad (r < 1),$$

then, for $r \leqslant 1$,

$$M_q(r,f) < K(\alpha, p, q)(1-r)^{-\alpha-1/p+1/q}, \quad \text{if} \quad q \geqslant 1, \tag{3.31}$$

$$M_q(r,f) < K(\alpha, p)(1-r)^{-\alpha+1-1/p}, \quad \text{if} \quad q \leqslant 1. \tag{3.32}$$

The more powerful methods fail when $p \leqslant 1$ owing to the fact that the maximal theorem* of Hardy and Littlewood for $u(r, \theta)$ does not hold for $p \leqslant 1$. I do not know whether the result can be improved for $p < 1$. The result for $p = 1$ is the same as in Theorem II, and is the best possible.

Proof. Let $f(z) = \sum_{n=1}^{\infty} c_n z^n$; then there is a θ_n such that

$$|c_n| R^n = \left| \frac{1}{\pi} \int_0^{2\pi} u(R, \theta) \cos(n\theta - \theta_n)\, d\theta \right| \leqslant M_1(R, u).$$

But, by (1.14), we have

$$M_1(R, u) \leqslant 2M_1(R, u_+) \leqslant 2M^{1-p}(R) M_p^p(R, u_+)$$
$$\leqslant 2A(1-R)^{-p\alpha} M^{1-p}(R).$$

If $r < R$, we have

$$M(r) \leqslant \sum_{n=1}^{\infty} |c_n| R^n \left(\frac{r}{R}\right)^n \leqslant \frac{2AR}{R-r} (1-R)^{-p\alpha} M^{1-p}(R).$$

It follows, as in the work of Hardy and Littlewood,† that

$$M(r) < K(\alpha, p) A (1-r)^{-\alpha-1/p};$$

and so

$$M_1(r, u) \leqslant 2A(1-r)^{-p\alpha} M^{1-p}(r) < K(\alpha, p) A (1-r)^{-\alpha+1-1/p}.$$

The inequality (3.31) now follows from Lemmas 2 and 3, and (3.32) from Hölder's inequality.

4.1. If (1.15) is satisfied for $\alpha > 1$, we can use methods similar to that by which (1.13) was obtained. For we can ignore those regions where $|f| < A(1-r)^{-1}$, and in particular the regions where $f(z)$ behaves like $-(1-z)^{-\beta}$, $\beta \leqslant 1$. For such functions $u_+ = 0$ inside the unit circle while $|u|$ is large. If $\alpha \leqslant 1$, we have to allow for functions of this type, which makes things very much more difficult. The chief feature of the methods used for (1.11) and (1.12) is the subtraction

* See § 4.1, Lemma 4 and footnote.

† G. H. Hardy and J. E. Littlewood, *J. für Math.* 167 (1931), 409, 410.

of a function of the form $(1-z)^{-\beta}$, $\beta < 1$, and this is only effective for $M(r)$. The result for $\alpha > 1$ has already been proved in Theorem II for $p = 1$; for $p > 1$ the theorem depends on a maximal result which is easily deduced from one of Hardy and Littlewood.*

Suppose that $0 < \beta < \frac{1}{2}\pi$, and let $S_\beta(r,\theta)$ denote the kite-shaped area defined by drawing two lines through $re^{i\theta}$ at angles β with the radius vector and dropping perpendiculars upon them from the origin. We denote by $U(r,\theta,\beta) = U(\beta)$, $U_+(r,\theta,\beta)$, $U_-(r,\theta,\beta)$, $U_1(r,\theta,\beta)$, the upper bounds in $S_\beta(r,\theta)$ of $|u|$, u_+, u_-, and u_1 respectively, where $u_- = -u$ if $u < 0$ and $u_- = 0$ otherwise, and u_1 is a harmonic function which we shall define in a moment.

Lemma 4. *If $p > 1$, $r < 1$, then*

$$M_p\{r, U_+(\beta)\} \leqslant K(p,\beta)M_p(r,u_+).$$

The theorem of Hardy and Littlewood on which this lemma depends is: *If $p > 1$, $r < 1$, then*

$$M_p\{r, U(\beta)\} \leqslant K(p,\beta)M_p(r,u). \tag{4.11}$$

Proof of Lemma 4. Let

$$u_1(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u_+(R,\phi)P(R,r,\theta-\phi)\,d\phi.$$

Then $u_1(r,\theta)$ is a harmonic function regular for $r < R$; and, since $P(R,r,\theta-\phi) \geqslant 0$, it follows from Poisson's formula for $u(r,\theta)$ that

$$0 \leqslant u(r,\theta) \leqslant u_1(r,\theta).$$

Hence, by (4.11), we have

$$M_p\{R, U_+(\beta)\} \leqslant M_p\{R, U_1(\beta)\} \leqslant K(p,\beta)M_p(R,u_1)$$
$$= K(p,\beta)M_p(R,u_+).$$

For $u_1(R,\theta) \equiv u_+(R,\theta)$.

4.2. The kernel of the proof of Theorem I is a result similar to Lemma 1 in principle, but expressed from the opposite point of view. We show that, if $u_-(r,\theta)$ increases more rapidly than $\mathbf{R}(e^{i\theta}-z)^{-\alpha}$ in $S_\beta(r,\theta)$, where $\beta > \frac{1}{2}\pi/\alpha$, then $u_+(r,\theta)$ is approximately as large as $u_-(r,\theta)$. The underlying principle is the same as in the theorem that, if an integral function is of order ρ in an angle greater than π/ρ, it has a direction of Borel of order ρ in that angle.† For the function

* G. H. Hardy and J. E. Littlewood, *Acta Math.* 54 (1930), 108, Theorem 17.
† See H. Milloux, *Acta Math.* 52 (1928), 236.

considered here may be compared with the logarithm of an integral function which has no zeros in a certain angle.

LEMMA 5. *If* $\alpha > 1$, $\frac{1}{2}\pi/\alpha < \beta < \frac{1}{2}\pi$, $k > 1$, *and*

$$U_{-}(R,\theta,\beta) > k^{\alpha}U_{-}\{1-k(1-R),\theta,\beta\} \tag{4.21}$$

for some θ *and some* R *such that* $1-1/k < R < 1$, *then*

$$U_{+}\{1-\tfrac{1}{2}(1-R),\theta,\tfrac{1}{4}\pi+\tfrac{1}{2}\beta\} > k^{-\alpha}U_{-}(R,\theta,\beta), \tag{4.22}$$

provided that $k > k_0(\alpha,\beta)$, $R > R_0(k,\beta)$.

In order to show that this gives (1.16) we require

LEMMA 6. *Suppose that*

$$M_p\{R_0, U_{-}(\beta)\} < A_1, \tag{4.23}$$

where $R_0 < 1$, $p > 1$, $\beta < \frac{1}{2}\pi$, *and that*

$$M_p\{R, U_{-}(\beta)\} > HA_1(1-R)^{-\alpha}, \tag{4.24}$$

for some R *such that* $R_0 < R < 1$. *Then given any* $k \geqslant 2$, *we can find positive numbers* $B(\alpha,k)$, $R_1 > 1-k(1-R_1) > R_0$ *and a set* E *of* θ *such that*

$$M_p\{R_1, U_{-}(\beta), E\} = \left\{\frac{1}{2\pi}\int_E \{U_{-}(R_1,\theta,\beta)\}^p\,d\theta\right\}^{1/p} > BHA_1(1-R_1)^{-\alpha} \tag{4.25}$$

and

$$U_{-}(R_1,\theta,\beta) > k^{\alpha}U_{-}\{1-k(1-R_1),\theta,\beta\} \tag{4.26}$$

for all θ *belonging to* E, *provided that* $H > H_0(k)$.

Proof of Theorem I for $\alpha > 1$. It follows from Theorem II and Lemma 4 that (1.15) implies (4.23) with $A_1 = K(\alpha, R_0, p, \beta)A$. Suppose that $\frac{1}{2}\pi/\alpha < \beta < \frac{1}{2}\pi$, and choose k and R_0 so that (4.21) implies (4.22) for $R > R_0$. Suppose that

$$M_p(R, u_{-}) > HA_1(1-R)^{-\alpha},$$

where $R_0 < R < 1$ and $H > H_0(k)$. Then (4.24) holds, and so (4.25) and (4.26) hold. Hence, by Lemma 5, we have

$$\begin{aligned} M_p\{1-\tfrac{1}{2}(1-R_1), U_{+}(\tfrac{1}{4}\pi+\tfrac{1}{2}\beta)\} &\geqslant M_p\{1-\tfrac{1}{2}(1-R_1), U_{+}(\tfrac{1}{4}\pi+\tfrac{1}{2}\beta), E\} \\ &> K(\alpha,k,\beta)M_p\{R_1, U_{-}(\beta), E\} \\ &> K(\alpha,k,\beta)BHA_1(1-R_1)^{-\alpha} \\ &= K(\alpha,k,\beta,p,R_0)HA(1-R_1)^{-\alpha}. \end{aligned}$$

Then, given any $k \geqslant 2$, we can find positive numbers $B(\alpha,k)$, $R_1 > \dots$.

Using Lemma 4, we have

$$M_p\{1-\tfrac{1}{2}(1-R_1), u_+\} > K(p,\beta)M_p\{1-\tfrac{1}{2}(1-R_1), U_+(\tfrac{1}{4}\pi+\beta)\}$$
$$> K(\alpha, k, \beta, p, R_0)HA(1-R_1)^{-\alpha}.$$

But this contradicts (1.15) when H is sufficiently large. Hence

$$M_p(R, u_-) < K(\alpha, p, R_0)A(1-R)^{-\alpha}$$

for $R_0 < R < 1$. Since R_0 depends only on β and k, which, in its turn, depends only on α and β, putting $\beta = \frac{1}{4}\pi(1+1/\alpha)$, we have (1.16) for $\alpha > 1$.

4.3. *Proof of Lemma* 6. Suppose that the result is not true. Let $r_1 = 1-(1-R_0)k^{-1}$, $r_2 = 1-(1-R_0)k^{-2}$,..., $r_M = 1-(1-R_0)k^{-M} \geqslant R$; and let E_n be the set of θ for which

$$U_-(r_n, \theta, \beta) > k^\alpha U_-(r_{n-1}, \theta, \beta).$$

Then $U_-(r_n, \theta, \beta) \leqslant k^\alpha U_-(r_{n-1}, \theta, \beta)$ in the complementary set CE_n. We suppose that

$$M_p\{r_n, U_-(\beta), E_n\} \leqslant B(\alpha, k)HA_1(1-r_n)^{-\alpha},$$

where $B(\alpha, k)$ is independent of k and n, for $n = 1, 2,..., M$ and all $k \geqslant 2$, where $H > H_0(k)$. Then

$$\begin{aligned}
M_p\{R, U_-(\beta)\} &\leqslant M_p\{r_M, U_-(\beta)\} \\
&\leqslant k^\alpha M_p\{r_{M-1}, U_-(\beta), CE_M\} + B(\alpha, k)HA_1(1-r_M)^{-\alpha} \\
&\leqslant k^{2\alpha} M_p\{r_{M-2}, U_-(\beta), CE_M\, CE_{M-1}\} + \\
&\qquad + B(\alpha, k)HA_1\{(1-r_M)^{-\alpha} + (1-r_{M-1})^{-\alpha}\} \\
&\leqslant \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&< k^{M\alpha} M_p\{R_0, U_-(\beta), CE_M\, CE_{M-1} \dots CE_1\} + \\
&\qquad + B(\alpha, k)HA_1(1-r_M)^{-\alpha}(1 + k^{-\alpha} + k^{-2\alpha} + \dots) \\
&< k^{M\alpha} M_p\{R_0, U_-(\beta)\} + B(\alpha, k)HA_1 \frac{(1-r_M)^{-\alpha}}{1-k^{-\alpha}} \\
&< A_1\left(\frac{1-r_M}{1-R_0}\right)^{-\alpha} + \frac{B(\alpha, k)}{1-2^{-\alpha}} HA_1(1-r_M)^{-\alpha} \\
&< A_1 k^\alpha \left(\frac{1-R}{1-R_0}\right)^{-\alpha} + \frac{B(\alpha, k)}{1-2^{-\alpha}} HA_1 k^\alpha (1-R)^{-\alpha},
\end{aligned}$$

where $CE_M\, CE_{M-1}$ denotes the product of CE_M and CE_{M-1}. But this contradicts (4.24), if $B(\alpha, k)$ is sufficiently small and $H > H_0(k)$. Hence (4.25) and (4.26) are satisfied for $E = E_n$, $R_1 = r_n$ for some $n \leqslant M$, provided that $H > H_0(k)$, which gives the required result.

4.4. Before proving Lemma 5, we need one further lemma.

LEMMA 7. *Suppose that $R < 1$, $\beta < \frac{1}{2}\pi$, and that $u_-(r,\theta)$ attains its upper bound $U_-(R,0,\beta)$ in $S_\beta(R,0)$ at $z' = r'e^{i\theta'}$. Then, for some ρ such that*

$$|1-\tfrac{1}{2}(1-R)-z'|-\tfrac{1}{2}(1-R)\sin\beta < \rho < |1-\tfrac{1}{2}(1-R)-z'|+\tfrac{1}{2}(1-R)\sin\beta,$$

we have

$$\int_{-\beta}^{\beta} u_-\{1-\tfrac{1}{2}(1-R)-\rho e^{i\phi}\}\rho\, d\phi > \tfrac{1}{4}\pi(1-R)\sin\beta\, U_-(R,0,\beta). \quad (4.41)$$

Proof. Writing $u(re^{i\theta})$ instead of $u(r,\theta)$, we have

$$-U_-(R,0,\beta) = u(r',\theta') = \frac{1}{2\pi}\int_0^{2\pi} u(z'+\rho' e^{i\phi'})\, d\phi',$$

where $\rho' \leqslant 1-R$, and so, integrating with respect to ρ', we have

$$\int_{\frac{1}{4}(1-R)\sin\beta}^{\frac{1}{2}(1-R)\sin\beta} \int_0^{2\pi} u_-(z'+\rho' e^{i\phi'})\, d\rho' d\phi' \geqslant \tfrac{1}{2}\pi(1-R)\sin\beta\, U_-(R,0,\beta).$$

But the left-hand side is equal to

$$\iint u_-\{1-\tfrac{1}{2}(1-R)-\rho e^{i\phi}\}\frac{\rho}{\rho'}\, d\rho d\phi$$

taken over points $\rho e^{i\phi}$ for which

$$\tfrac{1}{4}(1-R)\sin\beta \leqslant |1-\tfrac{1}{2}(1-R)-\rho e^{i\phi}-z'| \leqslant \tfrac{1}{2}(1-R)\sin\beta. \quad (4.42)$$

Since $\rho' < \frac{1}{2}(1-R)\sin\beta$ in that region, we have

$$\iint u_-\{1-\tfrac{1}{2}(1-R)-\rho e^{i\phi}\}\rho\, d\rho d\phi > \tfrac{1}{4}\pi(1-R)^2\sin^2\beta\, U_-(R,0,\beta);$$

and, since the variation in ρ is at most $(1-R)\sin\beta$, we have

$$\int u_-\{1-\tfrac{1}{2}(1-R)-\rho e^{i\phi}\}\rho\, d\phi > \tfrac{1}{4}\pi(1-R)\sin\beta\, U_-(R,0,\beta)$$

for some ρ in the range, and *a fortiori* within the limits stated, provided that the integral is taken over all values of ϕ for which (4.42) holds. Since these values lie within

$$|\arg\{1-\tfrac{1}{2}(1-R)-z'\}| \leqslant \beta,$$

we have (4.41).

4.5. *Proof of Lemma* 5. We may suppose without loss of generality that $\theta = 0$. Suppose that

$$U_+\{1-\tfrac{1}{2}(1-R), 0, \tfrac{1}{4}\pi+\tfrac{1}{2}\beta\} \leqslant k^{-\alpha}U_-(R,0,\beta). \quad (4.51)$$

Now, if $R > K(k,\beta)$, the circle

$$|z-1+k(1-R)| \leqslant (k-\tfrac{1}{2})(1-R)\sin\beta$$

lies in $S_\beta\{1-\frac{1}{2}(1-R), 0\}$, and includes the arc

$$|z-1+\tfrac{1}{2}(1-R)| = (k-\tfrac{1}{2})(1-R)\cos\beta, \quad |\arg\{z-1+\tfrac{1}{2}(1-R)\}| \leqslant \beta$$

for every $\beta < \frac{1}{2}\pi$. Applying Carathéodory's theorem to

$$f(z)-f\{1-k(1-R)\}$$

in the circle $|z-1+k(1-R)| \leqslant (k-\frac{1}{2})(1-R)\sin(\frac{1}{4}\pi+\frac{1}{2}\beta)$, we have, by (4.51),

$$|f(z)-f\{1-k(1-R)\}| \leqslant \frac{2k^{-\alpha}U_{-}(R,0,\beta)\sin(\frac{1}{4}\pi+\frac{1}{2}\beta)}{\sin(\frac{1}{4}\pi+\frac{1}{2}\beta)-\sin\beta'}$$

for $|z-1+k(1-R)| \leqslant (k-\frac{1}{2})(1-R)\sin\beta'$; and so, putting

$$\beta' = \tfrac{1}{8}\pi+\tfrac{3}{4}\beta,$$

we have, by (4.21),

$$|f(z)-v\{1-k(1-R), 0\}| \leqslant k^{-\alpha}U_{-}(R,0,\beta)K(\beta) \tag{4.52}$$

on the arc

$$|z-1+\tfrac{1}{2}(1-R)| = (k-\tfrac{1}{2})(1-R)\cos(\tfrac{1}{8}\pi+\tfrac{3}{4}\beta),$$

$$|\arg\{z-1+\tfrac{1}{2}(1-R)\}| \leqslant \tfrac{1}{8}\pi+\tfrac{3}{4}\beta.$$

Let ρ be a number for which (4.41) holds, and apply Carleman's formula* to

$$F_1(z) = U_1(z)+iV_1(z) = f\{1-\tfrac{1}{2}(1-R)-1/z\}-v\{1-k(1-R), 0\}$$

with $R = \rho^{-1}$, $l = (k-\frac{1}{2})(1-R)\cos\beta'$, $\beta = \beta' = \frac{1}{8}\pi+\frac{3}{4}\beta$, where the last β is that of (4.21). We have

$$\begin{aligned} I_1 &= \frac{l}{2\pi}\int\limits_{-\beta'}^{\beta'}\Big\{U_1(l,\theta)(l^{-\pi/2\beta'-1}+\rho^{\pi/2\beta'+1})\cos\frac{\theta\pi}{2\beta'} - \\ &\qquad\qquad -V_1(l,\theta)(l^{-\pi/2\beta'-1}-\rho^{\pi/2\beta'+1})\sin\frac{\theta\pi}{2\beta'}\Big\}\,d\theta \\ &= \frac{\rho^{\pi/2\beta'}}{\pi}\int\limits_{-\beta'}^{\beta'} U_1\Big(\frac{1}{\rho},\theta\Big)\cos\frac{\theta\pi}{2\beta'}\,d\theta + \\ &\qquad + \frac{1}{2\pi}\int\limits_{l}^{1/\rho}\{U_1(r,-\beta)+U_1(r,\beta)\}\{r^{-\pi/2\beta'}-(r\rho)^{\pi/2\beta'}\}\frac{dr}{r} \\ &= I_2+I_3. \end{aligned} \tag{4.53}$$

It follows from (4.42) that

$$|I_1| \leqslant k^{-\alpha}U_{-}(R,0,\beta)K(\beta)\{(k-\tfrac{1}{2})(1-R)\cos\beta'\}^{\pi/2\beta'}; \tag{4.54}$$

* See § 2.1.

and from (4.51), it follows that

$$I_3 < k^{-\alpha} U_-(R, 0, \beta) \frac{1}{\pi} \int_{l}^{1/\rho} r^{-\pi/2\beta'-1}\, dr \tag{4.55}$$

$$< k^{-\alpha} U_-(R, 0, \beta) K(\beta) \{(k-\tfrac{1}{2})(1-R)\cos\beta'\}^{\pi/2\beta'}.$$

On the other hand, by Lemma 7, we have

$$I_2 < -\cos\frac{\beta\pi}{2\beta'} \frac{\rho^{\pi/2\beta'}}{\pi} \int_{-\beta}^{\beta} u_-\{1-\tfrac{1}{2}(1-R)-\rho e^{i\phi}\}\, d\phi$$

$$< -K(\beta)\rho^{\pi/2\beta'-1}(1-R)\sin\beta\, U_-(R, 0, \beta). \tag{4.56}$$

Since $\beta' = \frac{1}{8}\pi + \frac{3}{4}\beta > \beta > \frac{1}{2}\pi/\alpha$, we have $\alpha > \frac{1}{2}\pi/\beta'$; and so the right-hand sides of (4.54) and (4.55) tend to 0 as $k \to \infty$. Since $\beta' < \frac{1}{2}\pi$, and $\rho > \frac{1}{2}(1-R)(1-\sin\beta)$, combining (4.54), (4.55), and (4.56) with (4.53), we have a contradiction for $k > k_0(\alpha, \beta)$, $R > R_0(R, \beta)$. It follows that (4.22) holds.

ON VAN DER CORPUT'S METHOD (VI)

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1. THIS paper is a sequel to the first paper* of the series; but it applies to lattice-point problems rather than to the zeta-function, so that the reference to the zeta-function in the title has been omitted.

In the problem of the lattice-points in a circle (see § 5.1 of the paper referred to) we consider sums of the form

$$\sum_{\nu=p}^{q} e^{2\pi i f(\nu)} \qquad (p < q \leqslant 2p), \tag{1}$$

where $f(\nu) = \{(\mu^2+\nu^2)y\}^{\frac{1}{2}}$. The order of these sums depends on the nature of the derivatives of $f(\nu)$, and, in particular, on†

$$f^{(5)}(\nu) = \frac{15\mu^2\nu(3\mu^2-4\nu^2)\sqrt{y}}{(\mu^2+\nu^2)^{\frac{9}{2}}}. \tag{2}$$

For $\nu > \mu$, we have $|f^{(5)}(\nu)| > A\mu^2\nu^{-6}\sqrt{y}$, and it is this inequality which is used to obtain the result. We thus use the fact that the zeros of the polynomial in the numerator do not fall within the range $\nu > \mu$.

There are, however, other problems in which we should expect the same result, but in which the zeros of the corresponding polynomial cannot be avoided in this way. Consider, for example, the problem of the lattice-points in the ellipse

$$au^2+buv+cv^2 = x,$$

where a, b, c are integers, and $a > 0$, $c > 0$, $b^2 < 4ac$. Let $R(x)$ now denote the number of these lattice-points, let

$$R_1(x) = \int_0^x R(t)\,dt,$$

and let $r(n)$ be the number of integer solutions of

$$au^2+buv+cv^2 = n.$$

Then

$$R_1(x) = \frac{\pi x^2}{\sqrt{(4ac-b^2)}}+\frac{x\sqrt{(4ac-b^2)}}{2\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n}J_2\left\{4\pi\sqrt{\left(\frac{nx}{4ac-b^2}\right)}\right\},$$

* *Quart. J. of Math.* (Oxford), 2 (1931), 161–73.

† In the section referred to, for $f^{(4)}(\nu)$ read $f^{(5)}(\nu)$.

the proof of this being substantially the same as in the case of a circle. Here we have to consider sums of the form (1), but with

$$f(\nu) = 2\sqrt{\left\{\frac{a\mu^2+b\mu\nu+c\nu^2}{4ac-b^2}y\right\}}. \tag{3}$$

We obtain

$$f^{(5)}(\nu) = \frac{\frac{15}{16}\sqrt{\{(4ac-b^2)y\}}}{(a\mu^2+b\mu\nu+c\nu^2)^{\frac{9}{2}}}\mu^2(b\mu+2c\nu)\{12c(a\mu^2+b\mu\nu+c\nu^2)-7(b\mu+2c\nu)^2\}$$

Suppose, for example, that $a = 1$, $b = -1$, $c = 1$. Then

$$f^{(5)}(\nu) = \tfrac{15}{16}\sqrt{(3y)}\frac{\mu^2(2\nu-\mu)(5\mu-4\nu)(\mu+4\nu)}{(\mu^2-\mu\nu+\nu^2)^{\frac{9}{2}}},$$

and we cannot avoid the zeros of the numerator either by taking $\nu > \mu$, or (interchanging μ and ν) by taking $\nu < \mu$.

We shall now give a variant of the original method which deals with this difficulty. We merely use the fact that the numerator is a polynomial of a certain degree, and the question of the position of its zeros does not arise.

2. We first prove the following simple lemma.

LEMMA α. *For any real polynomial $\phi(x)$ of degree n, with coefficient of x^n unity,*

$$\int\limits_{\xi}^{\xi+\delta} |\phi(x)|\, dx > K(n)\delta^{n+1},$$

where $K(n)$ depends on n only, not on ξ, δ, or the coefficients in the polynomial.

Suppose first that all the zeros of $\phi(x)$ are real, say

$$\phi(x) = (x-x_1)...(x-x_n).$$

Divide the interval $(\xi, \xi+\delta)$ into $n+1$ equal parts. At least one of these parts, say the mth, has no zero of $\phi(x)$ in its interior. Thus, for some q,

$$x_1, x_2,..., x_q \leqslant \xi+\frac{m-1}{n+1}\delta, \qquad \xi+\frac{m}{n+1}\delta \leqslant x_{q+1},..., x_n.$$

Hence

$$\int\limits_{\xi}^{\xi+\delta} |\phi(x)|\, dx$$

$$> \int\limits_{\xi+\frac{m-1}{n+1}\delta}^{\xi+\frac{m}{n+1}\delta} |\phi(x)|\, dx = \int\limits_{\xi+\frac{m-1}{n+1}\delta}^{\xi+\frac{m}{n+1}\delta} (x-x_1)...(x-x_q)(x_{q+1}-x)...(x_n-x)\, dx$$

$$\geqslant \int_{\xi+\frac{m-1}{n+1}\delta}^{\xi+\frac{m}{n+1}\delta} \left(x-\xi-\frac{m-1}{n+1}\delta\right)^{q}\left(\xi+\frac{m}{n+1}\delta-x\right)^{n-q} dx$$

$$= \frac{q!(n-q)!}{(n+1)!}\left(\frac{\delta}{n+1}\right)^{n+1},$$

and the result follows.

If $\phi(x)$ has conjugate complex zeros $\alpha \pm i\beta$, we have

$$(x-\alpha)^2+\beta^2 \geqslant (x-\alpha)^2,$$

so that this case is reduced to the previous one.

3. Lemma β. *In the range (x_1, x_2), let $f'(x)$ be monotonic, and let $f''(x) = \phi(x)\psi(x)$, where $\phi(x)$ is a real polynomial of degree n with coefficient of x^n unity, and $|\psi(x)| \geqslant \lambda$. Then*

$$\int_{x_1}^{x_2} e^{2\pi i f(x)}\, dx = O\left(\lambda^{-\frac{1}{n+2}}\right),$$

the constants depending on n only.

Now $f'(x)$ vanishes at most once in the range, say at $x = c$. Then

$$\int_{c}^{x_2} e^{2\pi i f(x)}\, dx = \int_{c}^{c+\delta} + \int_{c+\delta}^{x_2} = O(\delta)+O\left\{\frac{1}{|f'(c+\delta)|}\right\}.$$

Now

$$|f'(c+\delta)| = \left|\int_{c}^{c+\delta} f''(x)\, dx\right| \geqslant \lambda \int_{c}^{c+\delta} |\phi(x)|\, dx > K\lambda\delta^{n+1},$$

by Lemma α. Hence

$$\int_{c}^{x_2} e^{2\pi i f(x)}\, dx = O(\delta)+O(\lambda^{-1}\delta^{-n-1}),$$

and similarly for the integral over (x_1, c). Taking $\delta = \lambda^{-\frac{1}{n+2}}$, the result follows. There are obvious modifications to be made, if $c+\delta \geqslant x_2$, $c-\delta \leqslant x_1$, or if $f'(x)$ does not vanish in (x_1, x_2).

4. Let S denote the sum (1) of § 1, where $f(\nu)$ is defined by (3) of § 1. By Lemma 3 of paper I

$$S = O\left(\frac{p}{\sqrt{\rho}}\right)+O\left\{\left(\frac{p}{\rho}\sum_{\nu_1=1}^{\rho-1}|S_1|\right)^{\frac{1}{2}}\right\},$$

where

$$S_1 = \sum_{\nu=p}^{q-\nu_1} e^{2\pi i\{f(\nu+\nu_1)-f(\nu)\}}.$$

Applying the lemma again to S_1, but with ρ^2 instead of ρ, we obtain

$$S_1 = O\left(\frac{p}{\rho}\right)+O\left\{\left(\frac{p}{\rho^2}\sum_{\nu_2=1}^{\rho^2-1}|S_2|\right)^{\frac{1}{2}}\right\},$$

where $$S_2 = \sum_{\nu=p}^{q-\nu_1-\nu_2} e^{2\pi i\{f(\nu+\nu_1+\nu_2)-f(\nu+\nu_1)-f(\nu+\nu_2)+f(\nu)\}}.$$

Hence $$\sum_{\nu_1=1}^{\rho-1}|S_1| = O(p)+O\left\{\frac{p^{\frac{1}{2}}}{\rho}\sum_{\nu_1=1}^{\rho-1}\left(\sum_{\nu_2=1}^{\rho^2-1}|S_2|\right)^{\frac{1}{2}}\right\},$$

and $$S = O\left(\frac{p}{\sqrt{\rho}}\right)+O\left[\frac{p^{\frac{1}{2}+\frac{1}{4}}}{\rho^{\frac{1}{2}+\frac{1}{4}}}\left\{\sum_{\nu_1=1}^{\rho-1}\left(\sum_{\nu_2=1}^{\rho^2-1}|S_2|\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right]. \tag{4}$$

Repeating the process once more, we obtain

$$S = O\left(\frac{p}{\sqrt{\rho}}\right)+O\left[\frac{p^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}}{\rho^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}}\left[\sum_{\nu_1=1}^{\rho-1}\left\{\sum_{\nu_2=1}^{\rho^2-1}\left(\sum_{\nu_3=1}^{\rho^4-1}|S_3|\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}\right], \tag{5}$$

where $$S_3 = \sum_{\nu=p}^{q-\nu_1-\nu_2-\nu_3} e^{2\pi i\, F(\nu)},$$

and $F(\nu) = f(\nu+\nu_1+\nu_2+\nu_3)-\sum f(\nu+\nu_1+\nu_2)+\sum f(\nu+\nu_1)-f(\nu)$.

5. The function $F(\nu)$ is approximately equal to $\nu_1\nu_2\nu_3 f^{(3)}(\nu)$. If it were exactly equal to it, we could apply our polynomial argument at once. To show that the difference is negligible, the following complex-variable argument seems to be the most convenient. It introduces an element quite foreign to the original method, which is purely 'real'; but, after all, all the functions with which we deal are analytic.

We have

$$F(\mu z) = \int_0^{\nu_1} dx_1 \int_0^{\nu_2} dx_2 \int_0^{\nu_3} f'''(\mu z+x_1+x_2+x_3)\, dx_3,$$

$$F''(\mu z) = \int_0^{\nu_1} dx_1 \int_0^{\nu_2} dx_2 \int_0^{\nu_3} f^{(5)}(\mu z+x_1+x_2+x_3)\, dx_3,$$

and hence

$$\begin{aligned}|F''(\mu z)-\nu_1\nu_2\nu_3 f^{(5)}(\mu z)| &= \left|\int_0^{\nu_1} dx_1 \int_0^{\nu_2} dx_2 \int_0^{\nu_3} \{f^{(5)}(\mu z+x_1+x_2+x_3)-f^{(5)}(\mu z)\}\, dx_3\right| \\ &\leqslant \nu_1\nu_2\nu_3(\nu_1+\nu_2+\nu_3)\max|f^{(6)}(w)|,\end{aligned}$$

where w varies along the straight line from μz to $\mu z+\nu_1+\nu_2+\nu_3$. Now

$$f^{(5)}(\mu z) = \frac{\frac{15}{16}\sqrt{\{(4ac-b^2)y\}}}{\mu^4}\,\frac{(b+2cz)\{12c(a+bz+cz^2)-7(b+2cz)^2\}}{(a+bz+cz^2)^{\frac{9}{2}}}.$$

Hence we can find a rectangle $\xi_1 \leqslant \mathbf{R}(z) \leqslant \xi_2$, $-\eta \leqslant \mathbf{I}(z) \leqslant \eta$, $(0 < \xi_1 < \xi_2 \leqslant 2\xi_1,\ \eta \leqslant \xi_1)$ throughout which $f^{(5)}(\mu z)$ is regular, and on the perimeter of which

$$|f^{(5)}(\mu z)| > \frac{K\sqrt{y}}{\mu^4\xi_1^6},$$

K depending on a, b, and c only.

Also, $$|f^{(6)}(\mu z)| < \frac{K\sqrt{y}}{\mu^5\xi_1^7} \text{ throughout the rectangle.}$$

Hence too, $$|f^{(6)}(w)| < \frac{K\sqrt{y}}{\mu^5\xi_1^7},$$

if $w = \mu z + w'$ where $|w'| < \frac{1}{2}\xi_1\mu$. This is true in the above case, if $3\rho^4 < \frac{1}{2}\xi_1\mu$. Then

$$\left|\frac{F''(\mu z)}{\nu_1\nu_2\nu_3} - f^{(5)}(\mu z)\right| < \frac{K\rho^4\sqrt{y}}{\mu^5\xi_1^7}.$$

Hence, if $\rho^4\mu^{-1}\xi_1^{-1}$ is sufficiently small, it follows from Rouché's theorem that $F''(\mu z)$ has the same number of zeros inside the rectangle as $f^{(5)}(\mu z)$. Suppose for the sake of argument that this number is 3, this being the least favourable case; and let the zeros of $F''(\mu z)$ be z_1, z_2, z_3. Then

$$G(z) = \frac{(z-z_1)(z-z_2)(z-z_3)}{F''(\mu z)}$$

is regular inside and on the rectangle. On the perimeter

$$|G(z)| < \frac{K\mu^4\xi_1^9}{\nu_1\nu_2\nu_3\sqrt{y}},$$

and hence this holds inside the rectangle also, and, in particular, on the real axis. Hence

$$|F''(\mu x)| > K\nu_1\nu_2\nu_3 y^{\frac{1}{2}}\mu^{-4}\xi_1^{-9}|(x-z_1)(x-z_2)(x-z_3)|.$$

Hence, if $\nu \geqslant \mu$,

$$|F''(\nu)| > K\nu_1\nu_2\nu_3 y^{\frac{1}{2}}\mu^{-7}\xi_1^{-9}|(\nu-\mu z_1)(\nu-\mu z_2)(\nu-\mu z_3)|.$$

If $p \leqslant \nu \leqslant 2p$, we may take $\xi_1 = Ap/\mu$, and then

$$|F''(\nu)| > K\nu_1\nu_2\nu_3 y^{\frac{1}{2}}\mu^2 p^{-9}|(\nu-\mu z_1)(\nu-\mu z_2)(\nu-\mu z_3)|.$$

6. We divide up the range of variation of ν in S_3 at the zeros of $F''(x)$ and $F'''(x)$. We thus obtain $O(1)$ partial sums, in each of which F'' and F''' are of constant sign. It is sufficient to consider the case where $F'' > 0$, $F''' > 0$. Let $p_1 \leqslant \nu \leqslant p_2$ be such an interval. Then

F'' is positive and steadily increasing, and F' is steadily increasing. If $F'(x)$ takes the value $m+\frac{1}{2}$ in the range, let it do so at $x = x_m$. Suppose that $p_1 \leqslant x_{m-2} < x_m \leqslant p_2$.

$$\sum_{x_{m-1}\leqslant \nu < x_m} e^{2\pi i F(\nu)} = \int\limits_{x_{m-1}}^{x_m} e^{2\pi i\{F(x)-mx\}}\, dx + O(1) = O\left\{\frac{1}{\sqrt{F''(x_{m-1})}}\right\} + O(1),$$

by Lemmas 1 and 2 of I. Now

$$(x_{m-1}-x_{m-2})F''(x_{m-1}) \geqslant \int\limits_{x_{m-2}}^{x_{m-1}} F''(x)\, dx = F'(x_{m-1})-F'(x_{m-2}) = 1.$$

Hence
$$\sum_{x_{m-1}\leqslant \nu < x_m} e^{2\pi i F(\nu)} = O\{\sqrt{(x_{m-1}-x_{m-2})}\}+O(1).$$

Let m_0 be the least value of m such that $x_{m-2} \geqslant p_1$. Then

$$\sum_{x_{m_0-1}\leqslant \nu \leqslant p_2} e^{2\pi i F(\nu)} = O\left\{\sum_m \sqrt{(x_{m-1}-x_{m-2})}\right\} + O\left(\sum_m 1\right)$$
$$= O\left[\left\{\sum_m (x_{m-1}-x_{m-2}) \sum_m 1\right\}^{\frac{1}{2}}\right] + O\left(\sum_m 1\right).$$

Now
$$\sum_m (x_{m-1}-x_{m-2}) \leqslant p_2 - p_1 \leqslant p.$$

Also the total variation of $F'(x)$ in (p_1, p_2) is

$$O\{(p_2-p_1)\max F''(x)\} = O\left(p\frac{\nu_1\nu_2\nu_3\mu^2\sqrt{y}}{p^6}\right) = O\left(\frac{\nu_1\nu_2\nu_3\mu^2\sqrt{y}}{p^5}\right),$$

so that the number of values of m in the above sums is of this order.

Hence
$$\sum_{x_{m_0-1}\leqslant \nu \leqslant p_2} e^{2\pi i F(\nu)} = O\left\{\frac{(\nu_1\nu_2\nu_3)^{\frac{1}{2}}\mu y^{\frac{1}{4}}}{p^2}\right\} + O\left(\frac{\nu_1\nu_2\nu_3\mu^2\sqrt{y}}{p^5}\right).$$

We have still to consider the part of the sum over (p_1, x_{m_0-1}). By Lemma 1 of I

$$\sum_{p_1 \leqslant \nu < x_{m_0-1}} e^{2\pi i F(\nu)} = \int\limits_{p_1}^{x_{m_0-2}} e^{2\pi i\{F(x)-(m_0-2)x\}}\, dx + \int\limits_{x_{m_0-2}}^{x_{m_0-1}} e^{2\pi i\{F(x)-(m_0-1)x\}}\, dx + O(1).$$

By Lemma β and the result of the last section, each integral on the right is of the form

$$O\left\{\left(\frac{p^9}{\nu_1\nu_2\nu_3\sqrt{y}\mu^2}\right)^{\frac{1}{5}}\right\}.$$

Altogether we obtain

$$S_3 = O\left\{\frac{(\nu_1\nu_2\nu_3)^{\frac{1}{2}}\mu y^{\frac{1}{4}}}{p^2}\right\} + O\left\{\frac{\nu_1\nu_2\nu_3\mu^2\sqrt{y}}{p^5}\right\} + O\left\{\left(\frac{p^9}{\nu_1\nu_2\nu_3\mu^2\sqrt{y}}\right)^{\frac{1}{5}}\right\}.$$

Inserting this in (5) of § 4, we obtain, since $\mu \leqslant 2p$,

$$S = O\left(p\rho^{-\frac{1}{2}}\right) + O\left(p^{\frac{3}{4}}y^{\frac{1}{32}}\rho^{\frac{7}{16}}\right) + O\left(p^{\frac{1}{2}}y^{\frac{1}{16}}\rho^{\frac{7}{8}}\right) + O\left(p^{\frac{11}{10}}\rho^{-\frac{7}{40}}\mu^{-\frac{1}{20}}y^{-\frac{1}{80}}\right).$$

The first two terms are of the same form if $\rho = p^{\frac{4}{15}}y^{-\frac{1}{30}}$, and then

$$S = O\left(p^{\frac{13}{15}}y^{\frac{1}{60}}\right)+O\left(p^{\frac{11}{15}}y^{\frac{1}{30}}\right)+O\left(p^{\frac{79}{75}}y^{-\frac{1}{150}}\mu^{-\frac{1}{20}}\right).$$

The second term can be merged in the first, if $y \leqslant p^8$, i.e. if $\rho \geqslant 1$. If $\rho < 1$, then $S = O(p) = O(p\rho^{-\frac{1}{2}})$ trivially, so that in any case we obtain

$$S = O\left(p^{\frac{13}{15}}y^{\frac{1}{60}}\right)+O\left(p^{\frac{79}{75}}y^{-\frac{1}{150}}\mu^{-\frac{1}{20}}\right).$$

We have to sum with respect to μ over $\mu \leqslant p$, so that, in the result, μ can be replaced by p. The last term can then be merged in the first, if $p^{41} < Ay^7$. Now, as in the circle problem, we have ultimately to take $x \leqslant y < Ax$, and

$$p \leqslant \sqrt{N} = \sqrt{[x^{1-2\alpha}]} \leqslant x^{\frac{7}{41}},$$

so that the required condition is satisfied. The analysis of § 5 also requires that $\rho^4 < Ap$, i.e. $p < Ax^2$, which is true. We have therefore proved that

$$S = O\left(p^{\frac{13}{15}}y^{\frac{1}{60}}\right).$$

The corresponding formula in the case of the circle is that at the bottom of p. 171 of paper I, with $k = 4$, $K = 8$, viz.

$$\sum_{\nu=a}^{b} e^{2\pi i\sqrt{\{(\mu^2+\nu^2)y\}}} = O\left(a^{\frac{4}{5}}\mu^{\frac{1}{15}}y^{\frac{1}{60}}\right)+O\left(a^{\frac{7}{8}+\frac{1}{5}}\mu^{-\frac{1}{15}}y^{-\frac{1}{60}}\right).$$

The second term on the right is found to be negligible, and the first term has to be summed over $\mu \leqslant 2a$, so that there is no loss in replacing μ by a. Since the a of paper I is equivalent to the p of this paper, the two results are seen to be equivalent. We therefore obtain the same result for the ellipse as for the circle.

SOME EXTENSIONS OF YOUNG'S CRITERION FOR THE CONVERGENCE OF A FOURIER SERIES

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1. In this paper we are concerned mainly with the summability of a Fourier series at a particular point by Cesàro means of negative order, and *a fortiori* with its convergence. The theorems discussed, however, cover all orders of summability.

Let $f(t)$ be integrable L, and periodic with period 2π, and let its Fourier series be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt) = \tfrac{1}{2}A_0+\sum_{n=1}^{\infty}A_n(t).$$

Let
$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\}$$
and, for $t>0$, write

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)\,du \qquad (\alpha>0),$$

$$\Phi_0(t) = \phi(t), \qquad \Phi_\alpha(t) = \Phi'_{\alpha+1}(t) \qquad (-1<\alpha<0),$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t) \qquad (\alpha>-1).$$

It is known† that, if $\alpha \geqslant 0$ and $|\phi_\alpha(t)| = O(1)$ $(C,1)$,

i.e.
$$\frac{1}{t}\int_0^t |\phi_\alpha(u)|\,du = O(1), \tag{1}$$

in an interval $(0,\eta)$, then the Fourier series of $f(t)$ is either summable $(C,\alpha+\delta)$ at the point $t=x$, for every $\delta>0$, or not summable (A). We first consider the extension of this result to the case $-1<\alpha<0$.

The Cesàro summability of a Fourier series for negative orders is not a 'local' property, for, if the only non-local hypothesis is the Lebesgue integrability of the function, we know only that a_n and b_n are $o(1)$, and they need not be of smaller order. But the above result holds, even when $\alpha+\delta<0$, if (1) is satisfied *throughout the interval* $(0,\pi)$.‡ Since obtaining these results, however, I have discussed the problem of localization with Dr. A. C. Offord, and it now appears

† See Bosanquet (**3**), 28, where further references to particular cases are given.

‡ In this case $A_n(x) = O(n^\alpha)$.

that, for the class of Fourier series for which $A_n(x) = o(n^\gamma)$ as $n \to \infty$ $(-1 < \gamma < 0)$, summability (C, γ) does in fact depend only on the properties of $f(t)$ in the neighbourhood of the point $t = x$. By employing our new conditions for summability more satisfactory results are obtained.†

Condition (1) may also be replaced by the more general form

$$\frac{1}{t}\int_0^t u^{-\alpha}|d\Phi_{\alpha+1}(u)| = O(1).\ddagger \tag{2}$$

This may be regarded as an extension of Young's condition§

$$\frac{1}{t}\int_0^t |d\{u\phi(u)\}| = O(1), \tag{3}$$

which corresponds logically to the case $\alpha = -1$. The condition actually obtained by putting $\alpha = -1$ in (2) is more general than (3), but does not itself imply (2) for $\alpha > -1$, unless $\phi(t)$ is bounded, in which case it is equivalent to (3).|| These and other conditions are discussed in the later paragraphs.

2. LEMMA 1. *If* $-1 \leqslant \alpha < 0$, $\beta > \alpha$, *and* (i) $\Phi_{\alpha+1}(t)$ *is of bounded variation in an interval* $(0, \eta)$, (ii) $\Phi_{\alpha+1}(+0) = 0$, *then* $\Phi_{\beta+1}(t)$ *is a Lebesgue integral*, $\Phi_{\beta+1}(+0) = 0$, *and, for almost all* t *in* $(0, \eta)$,

$$\Phi_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha-1}\, d\Phi_{\alpha+1}(u). \tag{4}$$

† See Bosanquet and Offord (**5**).

‡ An integral like (2) is to be interpreted in the first instance as $\lim_{\epsilon \to 0} \int_\epsilon^t$, where it is assumed that $\Phi_{\alpha+1}(u)$ is of bounded variation in every interval (ϵ, t). But when condition (2) holds, writing $\Phi_\epsilon^*(t) = \int_\epsilon^t u^{-\alpha}|\,d\Phi_{\alpha+1}(u)|$, we observe that, if $\alpha > -1$ and $0 < \epsilon < t$,

$$\int_\epsilon^t |d\Phi_{\alpha+1}(u)| = \int_\epsilon^t u^\alpha |u^{-\alpha}\, d\Phi_{\alpha+1}(u)| = t^\alpha \Phi_\epsilon^*(t) - \alpha\int_\epsilon^t u^{\alpha-1}\Phi_\epsilon^*(u)\, du = O(t^{\alpha+1})$$

uniformly in ϵ and t. Thus $\Phi_{\alpha+1}(u)$ is of bounded variation in $(0, t)$, and the integral (2) exists as an ordinary Lebesgue-Stieltjes integral.

§ See Hardy and Littlewood (**7**), where other references are given. They give a theorem which corresponds to the case $\alpha = -1$, $\eta = \pi$ of Theorem 1, and which includes the previous results of Young and Pollard. When (3) is satisfied throughout the interval $(0, \pi)$ we have $A_n(x) = O(\log n/n)$.

|| The function $\phi(t) = \log|t|$ satisfies (2) when $\alpha = -1$, but not when $\alpha > -1$. See, however, Lemma 8.

We have, by the consistency theorem for fractional integrals of positive order,†

$$\Gamma(\beta-\alpha)\Phi_{\beta+1}(t) = \int_0^t (t-u)^{\beta-\alpha-1}\Phi_{\alpha+1}(u)\,du$$

$$= \left[\frac{(t-u)^{\beta-\alpha}\Phi_{\alpha+1}(u)}{\beta-\alpha}\right]_0^t + \frac{1}{\beta-\alpha}\int_0^t (t-u)^{\beta-\alpha}\,d\Phi_{\alpha+1}(u)$$

$$= \int_0^t d\Phi_{\alpha+1}(u) \int_u^t (v-u)^{\beta-\alpha-1}\,dv$$

$$= \int_0^t dv \int_0^v (v-u)^{\beta-\alpha-1}\,d\Phi_{\alpha+1}(u),$$

by the analogue of Fubini's theorem for Stieltjes integrals, since $\int_0^\eta |d\Phi_{\alpha+1}(u)|$ is finite.‡ It follows that

$$\Phi_{\beta+1}(t) = \int_0^t \Phi_\beta(u)\,du,$$

where
$$\Phi_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha-1}\,d\Phi_{\alpha+1}(u)$$

for almost all t in $(0, \eta)$, which proves the lemma.

THEOREM 1. *If* $-1 < \alpha < 0$, $\beta > \alpha$, *and* (i) $A_n(x) = o(n^\beta)$ *as* $n \to \infty$, (ii) *condition* (2) *holds in an interval* $(0, \eta)$,§ *then the Fourier series of* $f(t)$ *is either summable* (C, β) *for* $t = x$ *or not summable* (A). *A necessary and sufficient condition for summability to the sum* s *is that* $\phi_\gamma(t) = o(1)$ *as* $t \to 0$, *where* $\gamma > \alpha+1$.

Proof. We first observe that conditions (i) and (ii) are still satisfied if $f(t)$ is replaced by $f(t)+C|t-x|^{-\alpha-1}$, but the corresponding Fourier

† See Bosanquet (**1**), 135.

‡ We have
$$\int_a^b df(x) \int_c^d \phi(x,y)\,dg(y) = \int_c^d dg(y) \int_a^b \phi(x,y)\,df(x)$$

whenever $\int_a^b |df(x)| \int_c^d |\phi(x,y)|\,|dg(y)|$ exists. This is easily deduced from Fubini's theorem by expressing the Stieltjes integrals as Lebesgue integrals. Cf. Hardy, Littlewood, and Pòlya (**8**), 152–3.

§ Condition (2) is also equivalent to $\int_0^t u^{\rho-1-\alpha}|\,d\Phi_{\alpha+1}(u)| = O(t^\rho)$, $\rho > 0$, or $\int_t^\eta u^{1-\rho-\alpha}|\,d\Phi_{\alpha+1}(u)| = O(t^{-\rho})$, $\rho > 0$, as is easily verified by integration by parts.

series can only be summable (A) at the point $t = x$ for one value of C.† There will therefore be no loss of generality, if we only establish the theorem for the case where $\Phi_{\alpha+1}(+0) = 0$. In these circumstances it follows from (ii) that $\Phi_{\alpha+1}(t) = O(t^{\alpha+1})$,‡ i.e. $\phi(t) = O(1)$ $(C, \alpha+1)$, and hence that the Fourier series is bounded (C).§ If then it is summable (A) to s it is also summable (C) to s,|| and hence $\phi(t) = o(1)$ (C).†† Since $\phi(t) = O(1)$ $(C, \alpha+1)$ it now follows that $\phi(t) = o(1)(C, \gamma)$, i.e. $\phi_\gamma(t) = o(1)$, for $\gamma > \alpha+1$.‡‡ Thus the necessary condition is established, and for the rest of the proof there will be no loss of generality if we suppose in particular that $\phi_1(t) = o(1)$.

Now Offord and I have shown§§ that sufficient conditions for summability (C, β) to s are that

(a) $A_n(x) = o(n^\beta)$ as $n \to \infty$, (b) $\phi_1(t) = o(1)$ as $t \to 0$,

$$(c)\quad \lim_{\delta\to 0}\ \overline{\lim_{\tau\to\infty}}\ \overline{\lim_{\omega\to\infty}} \left| \omega^{-\beta} \int\limits_{\tau/\omega}^{\delta} \phi(t) \frac{\sin(\omega t - \frac{1}{2}\beta\pi)}{t^{1+\beta}}\, dt \right| = 0. \tag{5}$$

It therefore remains to establish condition (c), which is, in fact, a consequence of hypothesis (ii).

By Lemma 1 we have, for almost all t in the interval $(0, \eta)$,

$$\phi(t) = \frac{1}{\Gamma(-\alpha)} \int\limits_0^t (t-u)^{-\alpha-1}\, d\Phi_{\alpha+1}(u).$$

Hence we can write, if $0 < \delta < \eta$,

$$I = \Gamma(-\alpha) \int\limits_{\tau/\omega}^{\delta} \phi(t) \frac{\sin(\omega t - \frac{1}{2}\beta\pi)}{t^{1+\beta}}\, dt$$

$$= \int\limits_{\tau/\omega}^{\delta} t^{-1-\beta} \sin(\omega t - \tfrac{1}{2}\beta\pi)\, dt \int\limits_0^t (t-u)^{-\alpha-1}\, d\Phi_{\alpha+1}(u)$$

$$= \int\limits_0^{\tau/\omega} d\Phi_{\alpha+1}(u) \int\limits_{\tau/\omega}^{\delta} (t-u)^{-\alpha-1} t^{-1-\beta} \sin(\omega t - \tfrac{1}{2}\beta\pi)\, dt +$$

$$+ \int\limits_{\tau/\omega}^{\delta} d\Phi_{\alpha+1}(u) \int\limits_u^{\delta} (t-u)^{-\alpha-1} t^{-1-\beta} \sin(\omega t - \tfrac{1}{2}\beta\pi)\, dt$$

† For the Fourier series of $|t-x|^{-\alpha-1}$ diverges to $+\infty$ at $t = x$. But if $\phi(t) = |t|^{-\alpha-1}$ we have $\Phi_{\alpha+1}(t) = \Gamma(-\alpha)$ and so $\phi_\alpha(t) = 0$ for all $t > 0$.

‡ See footnote ‡, p. 114.

§ See Hardy and Littlewood (**6**), where the analogue for summability is given. || Littlewood (**9**). †† Hardy and Littlewood (**6**).

‡‡ See Bosanquet (**2**), 103. The result is due to M. Riesz. §§ Loc. cit.

$$= \int_0^{\tau/\omega} J(\delta, \tau, \omega, u)\, d\Phi_{\alpha+1}(u) + \int_{\tau/\omega}^{\delta} K(\delta, \omega, u)\, d\Phi_{\alpha+1}(u),$$

$$= I_1 + I_2.$$

The inversion of the two parts of the repeated integral is justified by the analogue of Fubini's theorem, since the resulting integrals are easily seen to be absolutely convergent.

We next show that, for $0 < u < \tau/\omega$,

$$|J(\delta, \tau, \omega, u)| \leqslant A\tau^{-1-\beta}\omega^{1+\beta+\alpha}, \tag{6}$$

where A denotes some number independent of δ, τ, ω, and u. Writing

$$J = \int_{\tau/\omega}^{\delta} = \int_{\tau/\omega}^{(\tau+1)/\omega} + \int_{(\tau+1)/\omega}^{\delta} = J_1 + J_2,$$

we have, if $0 < u < \tau/\omega$,

$$|J_1| \leqslant \left(\frac{\tau}{\omega}\right)^{-1-\beta} \int_{\tau/\omega}^{(\tau+1)/\omega} (t-u)^{-\alpha-1}\, dt \leqslant A\tau^{-1-\beta}\omega^{1+\beta+\alpha},$$

and, by the second mean-value theorem,

$$|J_2| = \left(\frac{\tau+1}{\omega}\right)^{-1-\beta} \left(\frac{1}{\omega}\right)^{-\alpha-1} \left| \int_{(\tau+1)/\omega}^{\zeta} \sin(\omega t - \tfrac{1}{2}\beta\pi)\, dt \right| \dagger$$

$$\leqslant A\tau^{-1-\beta}\omega^{1+\beta+\alpha}.$$

This establishes (6). It follows from this and condition (ii)‡ that

$$\omega^{-\beta}|I_1| = O(\tau^{-1-\beta}\omega^{1+\alpha}) \int_0^{\tau/\omega} |d\Phi_{\alpha+1}(u)|$$

$$= O(\tau^{-1-\beta}\omega^{1+\alpha}) O\{(\tau/\omega)^{\alpha+1}\},$$

which is annihilated by the operation $\lim\limits_{\tau\to\infty} \overline{\lim\limits_{\omega\to\infty}}$, since $\beta > \alpha$.

We can show by a similar argument that

$$|K(\delta, \omega, u)| \leqslant Au^{-1-\beta}\omega^{\alpha} \quad (0 < u < \delta) \tag{7}$$

if, when $u+\omega^{-1} < \delta$, we write§

$$K(\delta, \omega, u) = \int_u^{u+\omega^{-1}} + \int_{u+\omega^{-1}}^{\delta}.$$

† Where $(\tau+1)/\omega < \zeta < \delta$.

‡ See footnotes ‡, p. 114, and §, p. 115.

§ If $u+\omega^{-1} \geqslant \delta$, the integral need not be split up and the argument is simpler.

It follows from this and condition (ii)† that

$$\omega^{-\beta}|I_2| = O(\omega^{\alpha-\beta}) \int_{\tau/\omega}^{\delta} u^{-1-\beta}|\,d\Phi_{\alpha+1}(u)|$$

$$= O(\omega^{\alpha-\beta})O\{(\tau/\omega)^{\alpha-\beta}\},$$

which is annihilated by the operation $\lim_{\tau\to\infty}\overline{\lim}_{\omega\to\infty}$, since $\beta > \alpha$. Thus (5) is established and the theorem proved.

3. It is natural to expect that the conclusion of Theorem 1 will hold if condition (2) is replaced by

$$\frac{1}{t}\int_0^t |\phi_\alpha(u)-\phi_{\alpha+1}(u)|\,du = O(1). \tag{8}$$

We next obtain this result for the class of Fourier series summable (C) at the point $t = x$.

It is convenient to introduce the following notation. For $t > 0$ let

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-u)^{\alpha-2}\{u\phi(u)-\Phi_1(u)\}\,du \qquad (\alpha > 1),$$

$$\Psi_1(t) = t\phi(t)-\Phi_1(t),$$

$$\Psi_\alpha(t) = \Psi'_{\alpha+1}(t) \qquad (0 < \alpha < 1),$$

$$\psi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Psi_\alpha(t) \qquad (\alpha > 0).$$

Then

$$\Psi_1(t) = \int_0^t u\,d\phi(u) \tag{9}$$

whenever this integral exists as a Cauchy-Stieltjes integral.‡ We use the following lemmas.

† See footnote §, p. 115.

‡ That is, in the sense $\lim_{\epsilon\to 0}\int_\epsilon^t$. For the value of this integral is

$$\lim_{\epsilon\to 0}\left\{[u\phi(u)]_\epsilon^t - \int_\epsilon^t \phi(u)\,du\right\}.$$

But, writing $\Phi_\eta^*(t) = \int_t^\eta u\,d\phi(u)$, we see that

$$t\int_t^\eta d\phi(u) = t\int_t^\eta \frac{1}{u}u\,d\phi(u) = -\Phi_\eta^*(t)+t\int_t^\eta \Phi_\eta^*(u)\frac{du}{u^2},$$

the modulus of which is annihilated by the operation $\lim_{\eta\to 0}\overline{\lim}_{t\to 0}$. Thus $t\phi(t) = o(1)$ as $t\to 0$, and we obtain (9) without having to assume that $|\phi(t)|$ is integrable, provided $\Phi_1(t)$ is interpreted in the Cauchy sense.

LEMMA 2. *If $\alpha > 0$, then*

$$\psi_\alpha(t) = \alpha\{\phi_{\alpha-1}(t) - \phi_\alpha(t)\} = t\phi'_\alpha(t). \tag{10}$$

This follows from the definitions, and has been employed elsewhere.† We now see that (8) can be written in the more general form

$$\frac{1}{t}\int_0^t u|d\phi_{\alpha+1}(u)| = O(1). \tag{11}$$

LEMMA 3. *If $\alpha > 0$, then*

$$\frac{1}{\alpha}\int_0^\eta |\phi_\alpha(u)|\,du \tag{12}$$

is a non-increasing function of α.

If (12) is finite for a certain α, and $\beta > \alpha$, we have, by the theory of fractional integrals,

$$\frac{1}{\beta}\int_0^\eta |\phi_\beta(u)|\,du = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\int_0^\eta u^{-\beta}\,du\left|\int_0^u (u-v)^{\beta-\alpha-1}\Phi_\alpha(v)\,dv\right|$$

$$\leqslant \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\int_0^\eta u^{-\beta}\,du\int_0^u (u-v)^{\beta-\alpha-1}|\Phi_\alpha(v)|\,dv$$

$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\int_0^\eta |\Phi_\alpha(v)|\,dv\int_v^\eta u^{-\beta}(u-v)^{\beta-\alpha-1}\,du$$

$$\leqslant \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\int_0^\eta |\Phi_\alpha(v)|v^{-\alpha}\,dv\int_1^\infty w^{-\beta}(w-1)^{\beta-\alpha-1}\,dw$$

$$= \frac{1}{\alpha}\int_0^\eta |\phi_\alpha(v)|\,dv.$$

LEMMA 4. *If $\alpha > 0$, then*

$$\frac{1}{\alpha}\int_0^\eta u\,|d\phi_\alpha(u)| \tag{13}$$

is a non-increasing function of α.

† Bosanquet (**4**), 13.

Suppose (13) is finite for a certain α, and $\beta > \alpha$. Then, by a variation of Lemma 2, (13) can be expressed in the form

$$\Gamma(\alpha) \int_0^\eta u^{-\alpha} \, |d\Psi_{\alpha+1}(u)|, \tag{14}$$

and since this is finite it follows, on integration by parts,† that, if $0 < t < \eta$,

$$\int_0^t |d\Psi_{\alpha+1}(u)| = O(t^\alpha).$$

Thus $\Psi_{\alpha+1}(t)$ is of bounded variation in $(0, \eta)$. Again, since (13) is finite it follows, on integration by parts,‡ that $t\phi_\alpha(t) = o(1)$ as $t \to 0$, and hence that $t\phi_{\alpha+1}(t) = o(1)$. Thus, by Lemma 2, $\Psi_{\alpha+1}(t) = o(t^\alpha)$, and so $\Psi_{\alpha+1}(+0) = 0$. We now repeat the argument of Lemma 3, with $d\Psi_{\alpha+1}(t)$ in place of $\Phi_\alpha(t)$, applying Lemma 1, with $\Psi_{\alpha+1}(t)$ in place of $\Phi_{\alpha+1}(t)$, and obtain

$$\Gamma(\beta) \int_0^\eta u^{-\beta} |\Psi_\beta(u)| \, du = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \int_0^\eta u^{-\beta} \, du \left| \int_0^u (u-v)^{\beta-\alpha-1} \, d\Psi_{\alpha+1}(v) \right|$$

$$\leqslant \Gamma(\alpha) \int_0^\eta u^{-\alpha} \, |d\Psi_{\alpha+1}(u)|.$$

This proves the lemma.

LEMMA 5. *If* $\alpha > -1$, *necessary and sufficient conditions that* (2) *should hold and* $\Phi_{\alpha+1}(+0) = 0$ *are that* (11) *should hold and* $\phi_\lambda(t) = O(1)$ *for some* λ.

We have already observed that, if (2) holds and $\Phi_{\alpha+1}(+0) = 0$, then $\phi_{\alpha+1}(t) = O(1)$. The necessity of (11) then follows from the inequality

$$\frac{1}{\alpha+1} \int_0^t u |\phi'_{\alpha+1}(u)| \, du \leqslant \int_0^t |\phi_\alpha(u)| \, du + \int_0^t |\phi_{\alpha+1}(u)| \, du$$

obtained from Lemma 2, or the analogous inequality with Stieltjes integrals.

Again, if $\phi_\lambda(t) = O(1)$, for some positive λ, we have $|\phi_\lambda(t)| = O(1)$ $(C, 1)$. The sufficiency of this and (11) now follows, by Lemmas 3 and 4, on repeated application of the inequality

$$\int_0^t |\phi_{\kappa-1}(u)| \, du \leqslant \int_0^t |\phi_\kappa(u)| \, du + \frac{1}{\kappa} \int_0^t u |\phi'_\kappa(u)| \, du,$$

† Cf. footnote ‡, p. 114. ‡ Cf. footnote ‡, p. 118.

or its Stieltjes analogue, the condition $\Phi_{\alpha+1}(+0) = 0$ being a consequence of $|\phi_{\alpha+1}(t)| = O(1)$ $(C, 1)$.

The following lemma is also worth stating. Its effect is that, in the proof of Theorem 1, Stieltjes integrals may be avoided without loss of generality if we prefer that course.

LEMMA 6. *If* (2) *holds and* $\Phi_{\alpha+1}(+0) = 0$ *for a certain* $\alpha > -1$, *then* (1) *holds, with* α *replaced by* β, *and* $\Phi_{\beta+1}(+0) = 0$, *for every* $\beta > \alpha$.

We now have

THEOREM 2. *If* $\beta > \alpha > -1$, *and* (i) $A_n(x) = o(n^\beta)$ *as* $n \to \infty$, (ii) *condition* (11) *holds in an interval* $(0, \eta)$, *then the Fourier series is summable* (C, β) *if it is summable* (C).

4. We next consider the relation of condition (3) to the above results. We first remark that (3) is equivalent to

$$\frac{1}{t}\int_0^t u\,|d\phi(u)| = O(1) \tag{15}$$

and

$$\phi(t) = O(1).\dagger \tag{16}$$

For (16) is an immediate consequence of (3), and the rest of the statement follows from the inequalities

$$\int_0^t u\,|d\phi(u)| \leqslant \int_0^t |d\{u\phi(u)\}| + \int_0^t |\phi(u)|\,du \tag{17}$$

and

$$\int_0^t |d\{u\phi(u)\}| \leqslant \int_0^t u\,|d\phi(u)| + \int_0^t |\phi(u)|\,du. \tag{18}$$

Condition (15) is the case $\alpha = -1$ of (11).

We have the following lemmas.

† It is perhaps relevant to recall the fact that, if $s_n = a_0+a_1+\ldots+a_n$ and $s_n^{-1} = s_n+na_n$, then $s_n = \dfrac{1}{n+1}\sum_0^n s_\nu^{-1}$, from which it follows that $s_n^{-1} = O(1)$ if and only if $na_n = O(1)$ and $s_n = O(1)$. Also $\dfrac{1}{n+1}\sum_0^n |s_\nu^{-1}| = O(1)$, if and only if $\dfrac{1}{n+1}\sum_0^n \nu|a_\nu| = O(1)$ and $s_n = O(1)$.

LEMMA 7. *If* (15) *holds, then* (11) *holds for every* $\alpha > -1$.

Writing
$$\Psi_\gamma^*(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-u)^{\gamma-1} u\, |d\phi(u)|,$$

we have $\Psi_\gamma^*(t) = O(t^\gamma)$ for $\gamma > 1$. Now observing that
$$\int_0^t u\, |d\phi(u)| = \int_0^t |d\Psi_1^*(u)|$$

and $\Psi_1^*(+0) = 0$,† we apply Lemma 1, with $\alpha = -1$ and $\Phi_{\alpha+1}(t)$ replaced by $\Psi_1^*(t)$, and get, for $\beta > -1$,
$$\int_0^t u|\phi'_{\beta+1}(u)|\, du = \int_0^t |\psi_{\beta+1}(u)|\, du,$$
$$\leqslant \Gamma(\beta+2) \int_0^t u^{-\beta-1} \Psi_{\beta+1}^*(u)\, du$$
$$= \Gamma(\beta+2) t^{-\beta-1} \Psi_{\beta+2}^*(t) + (\beta+1)\Gamma(\beta+2) \int_0^t u^{-\beta-2} \Psi_{\beta+2}^*(u)\, du$$
$$= t^{-\beta-1} O(t^{\beta+2}) + \int_0^t u^{-\beta-2} O(u^{\beta+2})\, du$$
$$= O(t) + O(t).$$

LEMMA 8. *If* (3) *holds, then* (2) *holds and* $\Phi_{\alpha+1}(+0) = 0$ *for every* $\alpha > -1$.

Since (3) is equivalent to (15) and (16), the result follows from Lemmas 5 and 7.

LEMMA 9. *Necessary and sufficient conditions that* (3) *should hold are that* (15) *should hold and* $\phi(t) = O(1)$ (C).

It only remains to demonstrate the sufficiency. Now, by Lemma 7, (15) implies (11) for every $\alpha > -1$. Hence, by Lemma 8, the conditions imply (2) for every $\alpha > -1$, and in particular for $\alpha = 0$. The result then follows from (18).

We now have

THEOREM 3. *If* $\beta > -1$, *and* (i) $A_n(x) = o(n^\beta)$ *as* $n \to \infty$, (ii) *condition* (15) *holds throughout an interval* $(0, \eta)$, *then the Fourier series is summable* (C, β) *if it is summable* (C).

† By (9). See footnote ‡, p. 118.

5. It is now clear that, by repeating the above reasoning, (8) could be replaced by

$$\frac{1}{t}\int_0^t |\psi_{\alpha+1}(u)-\psi_{\alpha+2}(u)|\,du = O(1), \tag{19}$$

and so on to higher orders of delicacy. In the general case (11) would be replaced by

$$\frac{1}{t}\int_0^t u\left|d\left(u\frac{d}{du}\right)^{\kappa-1}\phi_{\alpha+\kappa}(u)\right| = O(1) \tag{20}$$

for some integer κ, where $\alpha \geqslant -1$.

An example of a function satisfying (20) in the case $\alpha = -1$, $\kappa = \lambda$, but not for any smaller value of κ, is $\phi(t) = \{\log|t|\}^{\lambda}$, where λ is an integer.

Added Feb. 23, 1935. It is interesting to know how much can be deduced from condition (8), or (11), alone. When $\eta = \pi$ it may be shown by direct arguments that $nA_n = O(1)$ $(C, \alpha+1+\delta)$, for every positive δ, and, conversely, whenever $nA_n = O(1)$ (C), condition (8) must be satisfied for sufficiently large α.

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ON THE DIFFERENCE OF CONSECUTIVE PRIMES

By P. ERDŐS (*Budapest*)

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WE consider here the question of the intervals between two consecutive prime numbers. Let p_n denote the nth prime. Backlund* proved that, for any positive ϵ and an infinity of n,

$$p_{n+1}-p_n > (2-\epsilon)\log p_n.$$

Brauer and Zeitz† showed that $2-\epsilon$ could be replaced by $4-\epsilon$. Westzynthius‡ proved that for an infinity of n

$$p_{n+1}-p_n > \frac{2\log p_n \operatorname{logloglog} p_n}{\operatorname{loglogloglog} p_n};$$

and Ricci§ has just shown that this can be improved to

$$p_{n+1}-p_n > c\log p_n \operatorname{logloglog} p_n$$

for an infinity of n and with a certain constant c. By increasing the precision of Brauer and Zeitz's method, I shall prove

THEOREM I. *For a certain positive constant c_1 and an infinity of values of n,*

$$p_{n+1}-p_n > \frac{c_1 \log p_n \operatorname{loglog} p_n}{(\operatorname{logloglog} p_n)^2}.$$

We reduce our problem to the proof of the following theorem.

THEOREM II. *For a certain positive constant c_2, we can find $c_2 p_n \log p_n/(\operatorname{loglog} p_n)^2$ consecutive integers so that no one of them is relatively prime to the product $p_1 p_2 \cdots p_n$, i.e. each of these integers is divisible by at least one of the primes $p_1, p_2, \ldots, p_n$.*

* R. J. Backlund, 'Über die Differenzen zwischen den Zahlen, die zu den n ersten Primzahlen teilerfremd sind': *Commentationes in honorem Ernesti Leonardi Lindelöf*, Helsinki, 1929.

† A. Brauer u. H. Zeitz, 'Über eine zahlentheoretische Behauptung von Legendre': *Sitz. Berliner Math. Ges.* 29 (1930), 116–25; H. Zeitz, *Elementare Betrachtung über eine zahlentheoretische Behauptung von Legendre* (Berlin 1930, Privatdruck).

‡ 'Über die Verteilung der Zahlen, die zu den n ersten Primzahlen teilerfremd sind', *Comm. Phys.-Math., Helsingfors*, (5) 25 (1931).

§ 'Ricerche aritmetiche sui polinomi II (Intorno a una proposizione non vera di Legendre)': *Rend. Circ. Mat. di Palermo*, 58 (1934).

We require some lemmas.

LEMMA 1. *Let m be any positive integer greater than 1, x and y any numbers such that $1 \leqslant x < y < m$, and N the number of primes p less than or equal to m such that $p+1$ is not divisible by any of the primes P, where $x \leqslant P \leqslant y$. Then*

$$N < \frac{c_3 m \log x}{\log m \log y},$$

where c_3 is a constant independent of m, x, and y.

We omit the proof since it is a direct application of the method of Brun.*

LEMMA 2. *If N_0 is the number of those integers not exceeding $p_n \log p_n$, each of whose greatest prime-factors is less than $p_n^{1/(20 \log\log p_n)}$, then $N_0 = o\{p_n/(\log p_n)^2\}$.*

We shall divide the integers we are considering into two classes: (i) those for each of which the number of different prime factors does not exceed $10 \log\log p_n$, and (ii) those for each of which the number of different prime factors exceeds $10 \log\log p_n$. Let the number of integers in these two classes be N_1 and N_2 respectively; then $N_0 = N_1 + N_2$.

If Q is a prime not exceeding $p_n^{1/(20 \log\log p_n)}$, then

$$Q^x > p_n \log p_n \quad \text{if} \quad x > (2 \log p_n)/(\log 2).$$

Hence the number of such primes and powers of such primes less than $p_n \log p_n$ is certainly less than

$$\frac{2 \log p_n}{\log 2} p_n^{1/(20 \log\log p_n)}.$$

But every integer of the class (i) is a product of not more than $10 \log\log p_n$ factors, each being one of these primes or powers. Hence

$$N_1 < \left(\frac{2 \log p_n}{\log 2} p_n^{1/(20 \log\log p_n)}\right)^{10 \log\log p_n}$$
$$= p_n^{\frac{1}{2}} \left(\frac{2 \log p_n}{\log 2}\right)^{10 \log\log p_n} = o\left\{\frac{p_n}{(\log p_n)^2}\right\}.$$

Let $d(k)$ be the number of divisors of k. If k is an integer of the second class, k has more than $10 \log\log p_n$ different prime factors and so

$$d(k) > 2^{10 \log\log p_n} > (\log p_n)^5.$$

* V. Brun, 'Le crible d'Ératosthène et le théorème de Goldbach': *Vidensk. Selsk. Skrifter, Mat.-naturv. Kl. Kristiania*, **3** (1920), and *Comptes Rendus*, **168** (1919). See also 'La série $\frac{1}{5}+\frac{1}{7}+\dots$ où les dénominateurs sont "nombres premiers jumeaux" est convergente ou finie', *Bull. Soc. Math.* (2) **43** (1919), 1–9.

Since
$$\sum_{l=1}^{p_n \log p_n} d(l) < 4p_n(\log p_n)^2$$
for sufficiently large n, we have
$$N_2 = o\left\{\frac{p_n}{(\log p_n)^2}\right\}.$$

LEMMA 3. *We can find a constant c_4 so that the number of primes p, less than $c_4 p_n \log p_n/(\log\log p_n)^2$ and such that $p+1$ is not divisible by any prime between $\log p_n$ and $p_n^{1/(20\log\log p_n)}$, is less than $p_n/4\log p_n$.*

We obtain this lemma immediately from Lemma 1 on putting
$$m = \frac{c_4 p_n \log p_n}{(\log\log p_n)^2}, \qquad x = \log p_n, \qquad y = p_n^{1/(20\log\log p_n)}.$$

We return now to Theorem II. We denote by q, r, s, t the primes satisfying the inequalities
$$1 < q \leqslant \log p_n, \qquad \log p_n < r \leqslant p_n^{1/(20\log\log p_n)},$$
$$p_n^{1/(20\log\log p_n)} < s \leqslant \tfrac{1}{2}p_n, \qquad \tfrac{1}{2}p_n < t \leqslant p_n.$$
We denote by $a_1, a_2,..., a_k$ the two sets of integers not greater than $p_n \log p_n$, namely (i) the prime numbers lying between $\frac{1}{2}p_n$ and $c_4 p_n \log p_n/(\log\log p_n)^2$ and not congruent to -1 to any modulus r, (ii) the integers not exceeding $p_n \log p_n$ whose prime factors are included only among the r. Some of the a's may be t's.

LEMMA 4. *The number of the t's is greater than k the number of the a's, if p_n is large enough.*

From Lemmas 2, 3,
$$k < \frac{1}{4}\frac{p_n}{\log p_n} + o\left\{\frac{p_n}{(\log p_n)^2}\right\}.$$
The number of the t's is greater than $\frac{1}{3}p_n/\log p_n$ for large p_n, as is evident from the prime-number theorem, and as can also be proved by elementary methods. This proves the lemma.

We now determine an integer z such that for all q, r, s,
$$0 < z < p_1 p_2 \cdots p_n,$$
$$z \equiv 0 \pmod{q}, \qquad z \equiv 1 \pmod{r}, \qquad z \equiv 0 \pmod{s},$$
$$z + a_i \equiv 0 \pmod{t_i} \qquad (i = 1, 2,..., k).$$

By Lemma 4, the last congruence is always possible, for, as there are more t's than a's, a case such as $z+a_1 \equiv 0 \pmod{t}$, $z+a_2 \equiv 0 \pmod{t}$ cannot occur.

We now show that, if l is any integer such that

$$0 < l < c_2 p_n \log p_n/(\log\log p_n)^2,$$

then no one of the integers

$$z,\ z+1,\ z+2,\ldots,\ z+l$$

is relatively prime to $p_1 p_2 \ldots p_n$.

Now any integer b $(0 < b < l)$ can be placed in one at least of the four following classes:

(i) $b \equiv 0 \pmod{q}$, for some q;
(ii) $b \equiv -1 \pmod{r}$, for some r;
(iii) $b \equiv 0 \pmod{s}$, for some s;
(iv) b is an a_i.

For b cannot be divisible by an r and by a prime greater than $\frac{1}{2}p_n$, since if this were so we should have

$$b > \tfrac{1}{2}p_n r > \tfrac{1}{2}p_n \log p_n > l,$$

for sufficiently large n. Hence, if b does not satisfy (i) or (iii), b is either a product of primes r only, and so satisfies (iv), or b is not divisible by any q, r, s. In the latter case, b must be a prime, for otherwise

$$b > (\tfrac{1}{2}p_n)^2 > l,$$

for sufficiently large n. Since, then, b is a prime between

$$\tfrac{1}{2}p_n \quad \text{and} \quad \frac{c_2 p_n \log p_n}{(\log\log p_n)^2},$$

b is either an a_i, or b satisfies (ii).

It is now clear that $z+b$ is not relatively prime to $p_1 p_2 \ldots p_n$, if

$$b < c_2 p_n \log p_n/(\log\log p_n)^2.$$

Hence also, if $p_1, p_2,\ldots, p_n$ are the primes not exceeding x, say, $z+b$ is not relatively prime to $p_1 p_2 \ldots p_n$, if $b < c_5 x \log x/(\log\log x)^2$, where c_3 is an appropriate constant independent of x. This is clear from the first case on noticing that, by Bertrand's theorem, $p_n \geqslant \frac{1}{2}x$.

We return to the main problem. Take $x = \frac{1}{2}\log p_n$. Then the product of the primes not exceeding x is less than $\frac{1}{2}p_n$ for large p_n by the prime-number theorem, or also by elementary methods. By Theorem II, since now $b < \frac{1}{2}p_n$, we can find K consecutive integers less than p_n, where

$$K = \frac{c_5 \log p_n \log\log p_n}{(\log\log\log p_n)^2},$$

each of which is divisible by a prime less than $\frac{1}{2}\log p_n$. Hence there

are at least $K-\frac{1}{2}\log p_n$ ($>\frac{1}{2}K$) consecutive integers which are not primes.

Thus we have proved that at least one of the intervals between successive primes less than p_n is always of length not less than $c\log p_n \operatorname{loglog} p_n/(\operatorname{logloglog} p_n)^2$ for large p_n and an appropriate constant c. Since this expression is an increasing function of n, it follows immediately that for an infinity of n,

$$p_{n+1}-p_n > \frac{c_1 \log p_n \operatorname{loglog} p_n}{(\operatorname{logloglog} p_n)^2}.$$

I wish to take this opportunity of expressing my gratitude to Professor Mordell for so kindly having helped me in preparing my manuscript.

A NOTE ON TWO SPECIAL PRIMALS IN FOUR DIMENSIONS

By J. A. TODD (*Manchester*)

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Introduction

THE determinantal quartic primal in [4], represented on [3] by quartic surfaces passing through a decimic curve of genus eleven, is well known.* It contains twenty double points, represented by the quadrisecant lines of the curve. If the curve C^{10} breaks up into simpler curves, the primal will, in general, contain additional nodes, for a point of intersection of two simple branches of the base curve maps, in general, a double point on the primal. For example, if the curve consists of two twisted cubics together with four of their common chords, the primal has in all 36 nodes, and has been described elsewhere.† The maximum number of extra nodes which can arise in this way is twenty, the curve in this case consisting of ten lines. We shall see that this can happen in two distinct ways, leading to two quartic primals with forty nodes. The object of this note is to describe briefly these primals and the configurations formed by their double points.

1. A set of ten lines forming a degenerate C^{10} of genus eleven has twenty mutual intersections. If the linear system of quartic surfaces through these lines maps a primal whose only singularities are ordinary double points, and which does not contain any planes, then it is not difficult to show that no three of the lines can be coplanar, that no six can lie on a quadric, that no one of the lines can meet as many as five of the others, and that no three of the lines can meet in a point. From this it follows that the set of ten lines is of one of the two following types:

(i) the ten lines form a 'double-five', i.e. consist of five lines $a_1,\ldots,a_5$ and five lines $b_1,\ldots,b_5$ where a_i meets b_j if $i \neq j$, and all other pairs of lines are skew;

(ii) the ten lines may be denoted by the symbols $1, 1_0,\ldots, 5, 5_0$, where i and i_0 both meet j and j_0 if $|i-j| \equiv 1 \pmod 5$, and all other pairs of lines are skew.

* L. Roth, *Proc. London Math. Soc.* (2) 30 (1930), 118–26.

† J. A. Todd, *Proc. Cambridge Phil. Soc.* 30 (1933), 52–68.

The possibility of 40-nodal primals containing planes will not be considered here. We proceed to discuss the two types mentioned above.

2. Consider the linear system of quartic surfaces passing through the lines of a double-five. The primal represented by the system possesses 40 nodes which are mapped as follows:

P_{i6},	represented by the transversal line of a_j $(j \neq i)$,
P_{6i},	represented by the transversal line of b_j $(j \neq i)$,
C_{ij6} $(\equiv C_{lmn})$,	represented by the transversal line of a_i, a_j, b_i, b_j,
P_{ij},	represented by the point $[a_i b_j]$.

There are likewise forty primes, whose intersection with the primal consists of a pair of quadric surfaces, the eighty such surfaces accounting for all the lines on the primal. These may be represented as follows:

π_{i6},	by the neighbourhood of b_i, and a F^4 having b_i double,
π_{6i},	by the neighbourhood of a_i, and a F^4 having a_i double,
γ_{ij6} $(\equiv \gamma_{lmn})$,	by the pair of quadrics $(a_i a_j b_l b_m b_n)$ and $(b_i b_j a_l a_m a_n)$,
π_{ij},	by the plane $(a_j b_i)$ and a cubic surface through the other lines.

From the representation it is easily verified that each prime contains twelve of the points, and each point lies in twelve of the primes, according to the following table of incidence.

π_{ij}	P_{ki} P_{jk} C_{ijk}
γ_{ijk}	P_{ij} P_{lm}

Cubic surfaces passing through $a_1 a_2 a_3 b_1 b_4 b_5$ form a homaloidal system and determine a symmetrical cubic Cremona transformation of space. Under this transformation the linear system of quartic surfaces is transformed into a similar system. Using accents to refer to the transformed system we note in particular that $\pi_{16} \rightarrow \gamma'_{456}$, $\pi_{46} \rightarrow \pi'_{51}$, $\pi_{45} \rightarrow \gamma'_{146}$. Taken with the table of incidence this shows that *the forty points and primes form a symmetrical and self-dual configuration.*

Two points (or primes) will be termed *syzygetic* or *azygetic* according as there exist four or two primes (or points) incident with both. Each point or prime is syzygetic with twenty-seven others, and azygetic with the remaining twelve. The relation is given, for the primes, in

the annexed table, and the corresponding relation for the points is obtained by changing from Greek to Latin letters.

	Syzygetic	*Azygetic*
π_{ij}	$\pi_{ji}\quad \pi_{ik}\quad \pi_{kj}\quad \pi_{lm}\quad \gamma_{ikl}$	$\pi_{ki}\quad \pi_{jk}\quad \gamma_{ijk}$
γ_{ijk}	$\pi_{il}\quad \pi_{li}\quad \gamma_{ijl}$	$\pi_{ij}\quad \pi_{lm}$

The expression of V_3^4 as a determinantal primal depends on the choice of five mutually syzygetic primes (such as π_{i6}) of which any three have two common nodes and any four have one common node. There are 216 pairs of such pentads, typified by

$$\begin{pmatrix}\pi_{16} & \pi_{26} & \pi_{36} & \pi_{46} & \pi_{56}\\ \pi_{61} & \pi_{62} & \pi_{63} & \pi_{64} & \pi_{65}\end{pmatrix},\quad \begin{pmatrix}\pi_{12} & \pi_{13} & \pi_{14} & \pi_{56} & \pi_{65}\\ \gamma_{256} & \gamma_{356} & \gamma_{456} & \pi_{51} & \pi_{61}\end{pmatrix},\quad \begin{pmatrix}\pi_{16} & \pi_{26} & \pi_{35} & \pi_{45} & \gamma_{346}\\ \pi_{52} & \pi_{51} & \pi_{64} & \pi_{63} & \gamma_{126}\end{pmatrix}.$$

Hence *the primal possesses* 216 *pairs of conjugate projective generations*.

The twelve nodes in any one of the primes lie on the C^4 in which the two quadrics lying in the prime intersect. On examining the points in (say) γ_{123} we find that they fall into four triads (P_{12}, P_{23}, P_{31}), (P_{21}, P_{32}, P_{13}), (P_{45}, P_{56}, P_{64}), (P_{54}, P_{65}, P_{46}). The twelve primes azygetic to γ_{123} each contain two points belonging to the same triad; the remaining primes each contain four points belonging to different triads. Thus any plane containing three points from three different triads contains a point of the fourth triad. Using a canonical elliptic parameter for the C^4 such that the sum of the parameters of four coplanar points is congruent to zero, we find for the twelve points the parameters $(\alpha_1, \alpha_1 \pm \theta)$, $(\alpha_2, \alpha_2 \pm \theta)$, $(\alpha_3, \alpha_3 \pm \theta)$, $(-\sum \alpha_i, -\sum \alpha_i \pm \theta)$, where $3\theta \equiv 0$. This shows further that the osculating planes at three points of a triad meet in the fourth point of intersection of the curve with the plane of the triad, and that the four points so obtained are coplanar.

It follows that the *group of automorphisms* of the configuration of points has order $40 \times 4! \times 3! \times 3 \times 3 = 51840$. We shall presently show that this group is simply isomorphic with the group of the twenty-seven lines on the general cubic surface.

There are thirty-six quadrics each of which contains thirty of the nodes, and ten of the C^4 on which they lie in sets of twelve. In fact, the C^4 in π_{12} imposes eight conditions on quadrics which contain it, that in π_{13} imposes four more, and that in π_{14} two further ones. The quadric determined by the fourteen conditions so imposed then contains the C^4 in the primes π_{52}, π_{53}, π_{54}, π_{62}, π_{63}, π_{64}, γ_{156}, since it

contains eight non-associated points of each curve. There are in all thirty-six of these quadrics. Each is typified by the set of ten C^4 it contains, or the set of ten nodes which it does not contain (which are easily verified to have the same symbols as the C^4). These quadrics, together with the primes containing the C^4 which lie on them, may be denoted as follows:

χ	γ_{ijk}
χ_{ijk}	π_{il} γ_{ijk}
$\chi_{ij,kl,mn}$	π_{ij} π_{ji} γ_{ikm}

Two of the quadrics have either four or one common C^4, each being related to fifteen of the others in the former way and to the remaining twenty in the latter. This suggests a correspondence between the quadrics and the Schur quadrics of a cubic surface. In fact, denoting the latter by Q, Q_{ij}, Q_{ijk},* we have the following correspondence, which establishes the identity of the groups of automorphisms of the two configurations.

$$\chi \to Q, \qquad \chi_{12,35,64} \to Q_{34},$$
$$\chi_{123} \to Q_{236}, \qquad \chi_{13,24,56} \to Q_{35},$$
$$\chi_{456} \to Q_{145}, \qquad \chi_{13,25,46} \to Q_{16},$$
$$\chi_{12,34,56} \to Q_{12}, \qquad \chi_{14,25,36} \to Q_{23}.$$

The rest of the correspondence follows from these relations, in virtue of the known relationships of the χ and Q.

The thirty nodes on any one of the quadrics form its intersection with the double lines of a Segre quartic primal; thus the joins of the three pairs of points (P_{ij}, P_{ji}), (P_{kl}, P_{lk}), (P_{mn}, P_{nm}) which lie on χ, meet in pairs, and so are concurrent.† The 36×15, i.e. 540, points of intersection of these lines in threes we shall call *S-points*; they are the diagonal points of the quadrangles formed by sets of four coplanar nodes. These lie by threes on 540 *S-lines* which are the joins of pairs of syzygetic nodes of the primal.

Consider now the following four S-points:

$$[(P_{12}P_{21}), (P_{34}P_{43}), (P_{56}P_{65})],$$
$$[(P_{13}P_{31}), (P_{25}P_{52}), (P_{46}P_{64})],$$
$$[(P_{13}P_{56}), (P_{43}P_{52}), (P_{46}P_{12})],$$
$$[(P_{31}P_{65}), (P_{34}P_{25}), (P_{64}P_{21})].$$

* See T. G. Room, *J. of London Math. Soc.* 7 (1932), 147–54.

† Compare my paper cited above, 54.

The first two are associated with χ, and do not lie on an S-line. The third is associated with $\chi_{15,36,42}$. Now the line $P_{13}P_{56}$ lies in the plane $P_{56}P_{65}P_{13}P_{31}$ which contains the first two S-points, similarly each of the lines $P_{43}P_{52}$, $P_{46}P_{12}$ lies in a plane through the first two S-points. Thus the third S-point lies on the line joining the other two, and in a similar way the fourth S-point lies on the same line. Thus the S-points lie by fours on 1080 new lines, *I-lines* say, each of which contains two pairs of S-points.

We refrain from further elaboration of the properties of this figure, though doubtless many more incidence relations exist. We merely make use of the last property to effect *a linear construction* of the figure. To do this we note, first, that the figure, like that of the double-five, must have five projective invariants, so that its freedom is 29. Accordingly we take an arbitrary Segre 15-point 15-line figure (freedom 24) whose edges (ij) are to be the joins $P_{ij}P_{ji}$; and fix the five points P_{12}, P_{13}, P_{14}, P_{24}, P_{65} on the appropriate edges. We know from what precedes that the S-lines $P_{ij}P_{kl}$, $P_{im}P_{nl}$ meet in an S-point on the I-line joining the S-points $[(P_{ij}P_{ji}), (P_{nl}P_{ln})], [(P_{kl}P_{lk}), (P_{im}P_{mi})]$. This I-line is known from the Segre figure. Hence it follows that a knowledge of any three of the four points P_{ij}, P_{im}, P_{kl}, P_{nl} enables the fourth to be constructed. It is easy to see that by continued application of this principle the five points already mentioned determine the remaining points P, and from these the ten points C are easily constructed.

We may remark, finally, that, if the ten lines of the double-five are so chosen that the points of any plane section lie on a cubic curve, the lines not lying on a cubic surface (a specialization easily seen to impose only one condition on the figure), the set of quartic surfaces through the lines determines an *involution** in space, and maps a *double quadric* whose surface of branch-points is of order eight, possesses forty nodes, and has forty singular primes which touch it along elliptic quartic curves. This surface might repay further investigation.

3. We now consider the primal represented by F^4 passing through ten lines forming a configuration of type (ii) above. We find again that the primal contains forty nodes. Twenty of these are mapped by transversals of sets of four of the ten lines such as $4\,4_0\,1\,2$; we

* Cf. D. Montesano, *Rend. Acc. Lincei* (4) 4_1 (1888), 207–15.

denote such a node by $(1\,2)'$; the other twenty correspond to the intersections of pairs of the lines, that corresponding to the intersection of 1 and 2 being denoted by $(1\,2)$. There are also forty primes whose intersection with the primal consists of a pair of quadrics, namely:

ten such as $\{1\}$, mapped by the neighbourhood of 1 and an F^4 having 1 double;

ten such as $\{1'\}$ mapped by the pair of quadrics $(44_0 55_0 1)$, $(22_0 33_0 1_0)$;

twenty such as $\{1\,2\}$ mapped by the plane of 1 and 2, and a cubic surface through the remaining lines.

By means of cubic transformations with base-curves selected from among the ten lines it is easily seen that the forty points and primes form a symmetrical system, but it is also easily seen that the configuration is *not* self-dual, although it is still the case that each point lies in twelve primes, and each prime contains twelve points. The table of incidences is:

(12)	$\{1\}$	$\{2\}$	$\{1'\}$	$\{2_0'\}$	$\{4'\}$	$\{4_0'\}$	$\{1\,2_0\}$	$\{1_0\,2\}$	$\{5\,1\}$	$\{5_0\,1\}$	$\{2\,3\}$	$\{2\,3_0\}$
$(12)'$	$\{1\}$	$\{2\}$	$\{4\}$	$\{4_0\}$	$\{1'\}$	$\{2_0'\}$	$\{1\,2\}$	$\{1_0\,2_0\}$	$\{5\,1_0\}$	$\{5_0\,1_0\}$	$\{2_0\,3\}$	$\{2_0\,3_0\}$

An examination in detail of this table brings to light the following facts:

(i) Two primes of the set have in common either two or four of the points, each prime being associated with twelve of the others in the former way and with the other twenty-seven in the latter. We call the two primes *azygetic* in the former case, and *syzygetic* in the second.

(ii) The primes fall into ten *quadruples*, namely $(\{1\}, \{1_0\}, \{1'\}, \{1_0'\})$, $(\{1\,2\}, \{1_0\,2\}, \{1\,2_0\}, \{1_0\,2_0\})$, etc., with the property that, if two primes are syzygetic or azygetic, the same relation holds between any other pair of primes belonging respectively to the same quadruples as the first.

(iii) Given one of the primes, the twelve which are azygetic to it pass in pairs through six chords of the C^4 lying in the prime. These chords lie in pairs in the three planes in which the prime is met by the remaining primes of the quadruple to which it belongs, and these planes pass through an edge of the self-polar tetrahedron of the curve. The twelve points lie in pairs on twelve lines, six through each of the two vertices of the tetrahedron that lie on the edge.

(iv) The two vertices of the tetrahedron so determined in each prime lie also in the three primes of the same quadruple, and belong to the tetrahedra in these. There are in all just five such points, each of the ten joining-lines being common to the four primes of a quadruple. We may denote the five points by $O_{12}, O_{23}, \ldots, O_{51}$, the prime $\{1\}$ containing O_{23}, O_{45}, and the prime $\{1\,2\}$ containing O_{45}, O_{34}. Each of the five points lies on sixteen lines containing a pair of nodes, the pairs in perspective with O_{23} (for instance) being

$(1\,2)$ $(1\,2_0)$,	$(1\,2)'$ $(1\,2_0)'$,	$(1_0\,2)$ $(1_0\,2_0)$,	$(1_0\,2)'$ $(1_0\,2_0)'$,
$(3\,4)$ $(3_0\,4)$,	$(3\,4)'$ $(3_0\,4)'$,	$(3\,4_0)$ $(3_0\,4_0)$,	$(3\,4_0)'$ $(3_0\,4_0)'$,
$(4\,5)$ $(4\,5_0)'$,	$(4_0\,5)$ $(4_0\,5_0)'$,	$(4\,5)'$ $(4\,5_0)$,	$(4_0\,5)'$ $(4_0\,5_0)$,
$(5\,1)$ $(5_0\,1)'$,	$(5_0\,1)$ $(5\,1)'$,	$(5\,1_0)$ $(5_0\,1_0)'$,	$(5_0\,1_0)$ $(5\,1_0)'$.

(v) The forty points fall into five sets of eight, e.g.

$$(1\,2), (1\,2_0), (1_0\,2), (1_0\,2_0), (1\,2)', (1\,2_0)', (1_0\,2)', (1_0\,2_0)',$$

each set consisting of two tetrahedra in fourfold perspective with centres at four of the points O (in the case considered these centres are all save O_{12}). Four of the primes pass through each edge of such a tetrahedron.

(vi) Given one of the points, the other thirty-nine consist of

(*a*) the thirty-two points not belonging to the same set of eight as the chosen one; through each of these pass three of the twelve primes through the given point;

(*b*) the three points belonging to the same set of eight, and to the same tetrahedron in this; through each such point pass four of the twelve primes;

(*c*) the four points of the other tetrahedron, each lying on six of the twelve primes.

By means of (v) we can give a linear construction for the figure. We take in [4] a simplex $O_{12}, \ldots, O_{51}$ and an arbitrary point $(1\,2)$ in the prime $O_{23}\,O_{34}\,O_{45}\,O_{51}$. The other points of the set containing $(1\,2)$ are then determined by incidences by the usual construction for desmic tetrads. Similarly, we select $(2\,3)$ arbitrarily in the prime $O_{12}\,O_{34}\,O_{45}\,O_{51}$ and construct the other seven points of the same set. The point $(3\,4)$ in the prime $O_{12}\,O_{23}\,O_{45}\,O_{51}$ is no longer arbitrary, since it lies in the prime $\{2\,3\}$, which contains the points $(2\,3_0)$, $(2_0\,3)$, $(2\,3)'$, $(2_0\,3_0)'$, $(1\,2)$, $(1_0\,2)$, $(1\,2_0)'$, $(1_0\,2_0)'$ already found. Thus $(3\,4)$ is constrained to lie in a definite plane. Similarly, $(4\,5)$ must lie on a

definite line, while (5 1) is completely fixed. In all, the freedom of the construction is $20+3+3+2+1$, or 29 as it should be.

If the five points O are taken as vertices of the simplex of reference, O_{34} being $(1, 0, 0, 0, 0)$, the sets of eight points may be taken to have coordinates

$$\pm(0, b_1, c_1, d_1, e_1), \qquad \pm(a_2, 0, c_2, d_2, e_2), \qquad \pm(a_3, b_3, 0, d_3, e_3),$$
$$\pm(a_4, b_4, c_4, 0, e_4), \qquad \pm(a_5, b_5, c_5, d_5, 0),$$

where the quantities are connected by relations of the type

$$a_2 b_3 c_1 = a_3 b_1 c_2, \qquad a_2 b_4 d_1 = a_4 b_1 d_2,$$

of which only six are independent. The involutory case arises, if the nodes all lie on a quadric, i.e. if

$$\begin{vmatrix} 0 & b_1^2 & c_1^2 & d_1^2 & e_1^2 \\ a_2^2 & 0 & c_2^2 & d_2^2 & e_2^2 \\ a_3^2 & b_3^2 & 0 & d_3^2 & e_3^2 \\ a_4^2 & b_4^2 & c_4^2 & 0 & e_4^2 \\ a_5^2 & b_5^2 & c_5^2 & d_5^2 & 0 \end{vmatrix} = 0.$$

A NOTE ON THE ZEROS OF $\zeta(s)$

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1. IN a recent paper* Titchmarsh uses the formula

$$f(t) = e^{i\vartheta}\zeta(\tfrac{1}{2}+it) = \sum_{n=1}^{k} \frac{\cos(\vartheta - t\log n)}{\sqrt{n}} + O(t^{-\frac{1}{4}}), \tag{1}$$

where $$k = [\sqrt{(t/2\pi)}]$$

and $$\vartheta = -\tfrac{1}{2}\operatorname{am}\chi(\tfrac{1}{2}+it) = -\tfrac{1}{2}\operatorname{am}\{\pi^{it}\Gamma(\tfrac{1}{4}-\tfrac{1}{2}it)/\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)\},$$

to prove some results about the distribution of the zeros of the zeta-function. The function $\vartheta(t)$ is a steadily increasing function of t, so that the equation $\vartheta(t) = \nu\pi$ (ν a positive integer) has just one solution t_ν, and t_ν is approximately a multiple of $\nu/\log\nu$.

In his paper Titchmarsh proves that $(-1)^\nu f(t_\nu)$ is positive on the average, and that $f(t_\nu)f(t_{\nu+1})$ is negative on the average. From the latter result it follows that, in an infinity of cases, the interval $(t_\nu, t_{\nu+1})$ contains a zero of $\zeta(\frac{1}{2}+it)$. Writing N' for the number of negative terms in the sum

$$\sum_{\nu=M+1}^{N} f(t_\nu)f(t_{\nu+1}),$$

in which M is regarded as fixed, he proves that

$$N' > AN^{\frac{3}{5}}\log^{-\frac{1}{5}}N, \tag{2}$$

$$AN < N'^{\frac{1}{2}}\Big[\sum_{\nu=M+1}^{N}\{f(t_\nu)f(t_{\nu+1})\}^2\Big]^{\frac{1}{2}}. \tag{3}$$

Calling $G(T)$ the number of intervals in $(0, T)$ for which $(t_\nu, t_{\nu+1})$ contains a zero of $\zeta(\frac{1}{2}+it)$, he proves from (2) that

$$G(T) > AT^{\frac{3}{5}}/\log T.$$

He then suggests that (3) would lead to a more precise result, if the order of the sum $\sum_{\nu=M+1}^{N}\{f(t_\nu)f(t_{\nu+1})\}^2$ could be found.

It is my object in this paper to prove that

$$\sum_{\nu=M+1}^{N}\{f(t_\nu)f(t_{\nu+1})\}^2 = \frac{3}{8\pi^4}N\log^4 N + O(N\log^{\frac{7}{2}}N).$$

* E. C. Titchmarsh, 'On van der Corput's method and the zeta-function of Riemann (IV)': *Quart. J. of Math.* (Oxford), 5 (1934), 98–105.

Using this result in (3) we shall have

$$AN < AN'^{\frac{1}{2}}N^{\frac{1}{2}}\log^2 N,$$

i.e.

$$N' > AN/\log^4 N,$$

i.e.

$$G(T) > AT/\log^3 T.$$

2. The method I use closely follows that of the paper quoted above. In addition to the elements of van der Corput's method, for which we refer to an earlier paper* of Titchmarsh, we shall require the following lemmas.

2.1. LEMMA 1. *If $f(x)$ is a real, differentiable function in the interval (a, b), $f'(x)$ is monotonic in this interval and $|f'(x)| < 1-\delta$, where $0 < \delta < 1$, then*

$$\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)}\,dx + O(1/\delta).$$

This is a slight extension of Lemma 1 of (I) and scarcely needs separate proof.

2.2. LEMMA 2. *As $x \to \infty$,*

$$\sum_1^x \frac{d(n)}{\sqrt{n}} = O(\sqrt{x}\log x), \qquad \sum_1^x \frac{d(n)}{\sqrt{(n\log n)}} = O\{\sqrt{(x\log x)}\},$$

$$\sum_1^x \frac{d(n)}{n\sqrt{(\log n)}} = O(\log^{\frac{3}{2}} x), \qquad \sum_1^x \frac{\{d(n)\}^2}{n\sqrt{(\log n)}} = O(\log^{\frac{7}{2}} x).$$

These all follow by partial summation from the well-known formulae†

$$\sum_1^x d(n) \sim x\log x,$$

$$\sum_1^x \{d(n)\}^2 = \frac{1}{\pi^2}x\log^3 x + O(x\log^2 x).$$

2.3. LEMMA 3. *As $x \to \infty$,*

$$\sum_{1<n<m<x}\sum \frac{d(m)d(n)}{\sqrt{(mn)}\log(m/n)} = O(x\log^3 x).$$

This is proved by Ingham.‡

* E. C. Titchmarsh, 'On van der Corput's method and the zeta-function of Riemann (I)': *Quart. J. of Math.* (Oxford), 2 (1931), 161–73. I shall refer to it simply as (I).

† The second of these is due to Ramanujan. See B. M. Wilson, 'Proofs of some formulae enunciated by Ramanujan': *Proc. London Math. Soc.* (2) 21 (1923), 235–55.

‡ A. E. Ingham, 'Mean-value theorems in the theory of the Riemann zeta-function': *Proc. London Math. Soc.* (2) 27 (1928), 297, Lemma B. 3.

2.4. I use the approximate functional equation for the square of the zeta-function, viz.*

$$\zeta^2(s) = \sum_{n\leqslant x}\frac{d(n)}{n^s}+\chi^2(s)\sum_{n\leqslant y}\frac{d(n)}{n^{1-s}}+O\left\{x^{\frac{1}{2}-\sigma}\left(\frac{x+y}{t}\right)^{\frac{1}{4}}\log t\right\},$$

where $-\frac{1}{2}\leqslant\sigma\leqslant\frac{3}{2}$, $x>A$, $y>A$, $xy=(t/2\pi)^2$, and $\chi(s)$ has its usual meaning. In the case $\sigma=\frac{1}{2}$ we have $\chi(\frac{1}{2}+it)=e^{-2i\vartheta}$, where ϑ is the same as in (1), and taking $x=y=t/2\pi$ we can obtain

$$\begin{aligned}\{f(t)\}^2 &\equiv e^{2i\vartheta}\zeta^2(\tfrac{1}{2}+it)\\ &= \sum_{n\leqslant t/2\pi}\frac{d(n)}{\sqrt{n}}\cos(2\vartheta-t\log n)+O(\log t),\end{aligned}$$

and

$$\begin{aligned}\{f(t_\nu)\}^2 &= \sum_{n\leqslant t_\nu/2\pi}\frac{d(n)}{\sqrt{n}}\cos(t_\nu\log n)+O(\log t_\nu)\\ &= h(t_\nu)+O(\log t_\nu),\quad\text{say.}\end{aligned}$$

3. For convenience of notation we sometimes write $\tau_\nu=t_\nu/2\pi$. We have

$$\begin{aligned}&\sum_{\nu=M+1}^{N}h(t_\nu)h(t_{\nu+1})\\ &= \sum_{\nu=M+1}^{N}\sum_{m\leqslant\tau_\nu}\frac{d(m)}{\sqrt{m}}\cos(t_\nu\log m)\sum_{n\leqslant\tau_{\nu+1}}\frac{d(n)}{\sqrt{n}}\cos(t_{\nu+1}\log n)\\ &= \sum_{m\leqslant\tau_N}\frac{d(m)}{\sqrt{m}}\sum_{n\leqslant\tau_{N+1}}\frac{d(n)}{\sqrt{n}}\sum_{\nu=\nu_1}^{N}\cos(t_\nu\log m)\cos(t_{\nu+1}\log n)\\ &= \frac{1}{2}\sum\frac{d(m)}{\sqrt{m}}\sum\frac{d(n)}{\sqrt{n}}\sum\cos(t_\nu\log m-t_{\nu+1}\log n)+\\ &\qquad+\frac{1}{2}\sum\frac{d(m)}{\sqrt{m}}\sum\frac{d(n)}{\sqrt{n}}\sum\cos(t_\nu\log m+t_{\nu+1}\log n)\\ &= \tfrac{1}{2}\textstyle\sum_1+\tfrac{1}{2}\sum_2,\quad\text{say,}\end{aligned}$$

where ν_1 is the least integer such that $\nu_1\geqslant M+1$, $\tau_{\nu_1}\geqslant m$, and $\tau_{\nu_1+1}\geqslant n$.

Now t_ν is defined by the equation

$$\vartheta(t_\nu)=\nu\pi,$$

whence

$$\vartheta'(t_\nu)\frac{dt_\nu}{d\nu}=\pi.$$

* See G. H. Hardy and J. E. Littlewood, 'The approximate functional equations for $\zeta(s)$ and $\zeta^2(s)$': *Proc. London Math. Soc.* (2) 29 (1929), 81–97.

Also

$$\vartheta'(t) = -\frac{1}{2}\frac{\chi'(\frac{1}{2}+it)}{\chi(\frac{1}{2}+it)} = -\frac{1}{2}\left\{\log\pi - \frac{1}{2}\frac{\Gamma'(\frac{1}{4}-\frac{1}{2}it)}{\Gamma(\frac{1}{4}-\frac{1}{2}it)} - \frac{1}{2}\frac{\Gamma'(\frac{1}{4}+\frac{1}{2}it)}{\Gamma(\frac{1}{4}+\frac{1}{2}it)}\right\}$$

$$= -\tfrac{1}{2}\log\pi + \tfrac{1}{4}\log(\tfrac{1}{16}+\tfrac{1}{4}t^2) - \frac{1}{1+4t^2} -$$

$$-\mathbf{R}\int_0^\infty \frac{u\,du}{\{u^2+(\frac{1}{4}+\frac{1}{2}it)^2\}(e^{2\pi u}-1)},$$

and it follows that, as $t \to \infty$,

$$\vartheta'(t) = \tfrac{1}{2}\log(t/2\pi) + O(1/t^2), \qquad \vartheta''(t) \sim 1/(2t).$$

3.1. Going back to $\sum_1$, we consider first the terms in which $m = n$. These contribute the sum

$$\sum_{\nu=M+1}^{N}\sum_{m\leqslant\tau_\nu}\frac{\{d(m)\}^2}{m}\cos\{(t_{\nu+1}-t_\nu)\log m\}.$$

We write $\lambda = t_{\nu+1}-t_\nu$ and

$$\phi(x) = -\frac{d}{dx}\left\{\frac{\cos(\lambda\log x)}{x}\right\}$$

$$= \frac{\lambda\sin(\lambda\log x)+\cos(\lambda\log x)}{x^2}.$$

If we also write $D(x) = \sum_1^x \{d(n)\}^2$, we shall have

$$\sum_{m\leqslant x}\frac{\{d(m)\}^2}{m}\cos(\lambda\log m) = \sum_1^{[x]}\frac{D(m)-D(m-1)}{m}\cos(\lambda\log m)$$

$$= \sum_1^{[x]-1} D(m)\left\{\frac{\cos(\lambda\log m)}{m} - \frac{\cos\{\lambda\log(m+1)\}}{m+1}\right\} + \frac{D(x)\cos(\lambda\log[x])}{[x]}$$

$$= \sum_1^{[x]-1} D(m)\int_m^{m+1}\phi(x)\,dx + \frac{D(x)\cos(\lambda\log[x])}{[x]}$$

$$= \int_1^x D(x)\phi(x)\,dx + \frac{D(x)\cos(\lambda\log x)}{x}.$$

Now, by Ramanujan's formula,* we have, as $x \to \infty$,

$$D(x) = \frac{1}{\pi^2}x\log^3 x + O(x\log^2 x).$$

* See B. M. Wilson, loc. cit.

Hence

$$\sum_{m\leqslant\tau_\nu}\frac{\{d(m)\}^2}{m}\cos(\lambda\log m) = \frac{1}{\pi^2}\int_1^{\tau_\nu}\{\lambda\sin(\lambda\log x)+\cos(\lambda\log x)\}\log^3 x\frac{dx}{x}+$$

$$+O\left\{\int_1^{\tau_\nu}(\lambda+1)\log^2 x\frac{dx}{x}\right\}+O(\log^3\tau_\nu). \qquad (4)$$

Now

$$\begin{aligned}\lambda = t_{\nu+1}-t_\nu &= \frac{\pi}{\vartheta'(t_\nu)}+O\left\{\frac{\vartheta''(t_\nu)}{\{\vartheta'(t_\nu)\}^3}\right\}\\ &= \frac{2\pi}{\log\tau_\nu+O(1/\tau_\nu^2)}+O\left(\frac{1}{\tau_\nu\log^3\tau_\nu}\right)\\ &= \frac{2\pi}{\log\tau_\nu}+O\left(\frac{1}{\tau_\nu\log^3\tau_\nu}\right).\end{aligned}$$

Thus the terms on the right of (4) reduce to

$$\frac{1}{\pi^2}\int_0^{\tau_\nu}\cos(\lambda\log x)\log^3 x\frac{dx}{x}+O(\log^3\tau_\nu).$$

If, in this integral, we put $\lambda\log x = y$, it becomes

$$\begin{aligned}\int_0^{\lambda\log\tau_\nu}\left(\frac{y}{\lambda}\right)^3\cos y\frac{dy}{\lambda} &= \frac{1}{\lambda^4}\left\{\int_0^{2\pi}y^3\cos y\,dy+O(2\pi-\lambda\log\tau_\nu)\right\}\\ &= \frac{\log^4\tau_\nu}{\{2\pi+O(1/\tau_\nu\log^2\tau)\}^4}\left\{12\pi^2+O\left(\frac{1}{\tau_\nu\log^2\tau_\nu}\right)\right\},\end{aligned}$$

and we obtain

$$\sum_{m\leqslant\tau_\nu}\frac{\{d(m)\}^2}{m}\cos\{(t_{\nu+1}-t_\nu)\log m\} = \frac{3}{4\pi^4}\log^4\tau_\nu+O(\log^3\tau_\nu).$$

It is easily seen that $\log\tau_\nu = \log\nu+O(\log\log\nu)$. Hence, for a fixed M,

$$\begin{aligned}&\sum_{\nu=M+1}^{N}\sum_{m\leqslant\tau_\nu}\frac{\{d(m)\}^2}{m}\cos\{(t_{\nu+1}-t_\nu)\log m\}\\ &\qquad = \frac{3}{4\pi^4}N\log^4 N+O(N\log^3 N\log\log N).\end{aligned}$$

3.2. In the remaining terms of $\sum_1$ the ν-sum is of the form

$$\sum\cos\{2\pi\psi(\nu)\},$$

where $$\psi(\nu) = \frac{1}{2\pi}(t_\nu\log m-t_{\nu+1}\log n).$$

Hence

$$\begin{aligned}\psi'(\nu) &= \frac{\log m}{2\vartheta'(t_\nu)} - \frac{\log n}{2\vartheta'(t_{\nu+1})} \\ &= \frac{\log m}{\log\tau_\nu + O(1/\tau_\nu^2)} - \frac{\log n}{\log\tau_{\nu+1} + O(1/\tau_{\nu+1}^2)} \\ &= \frac{(\log m - \log n)\log\tau_\nu + O\{(\log mn)/(\tau_\nu \log\tau_\nu)\}}{\log^2\tau_\nu\{1+O(1/\tau_\nu\log^2\tau_\nu)\}} \\ &= \frac{\log m - \log n}{\log\tau_\nu} + O\left\{\frac{\log mn}{\tau_\nu \log^3\tau_\nu}\right\}, \qquad (5)\end{aligned}$$

since $$\log\tau_{\nu+1} - \log\tau_\nu = O\left(\frac{\tau_{\nu+1}-\tau_\nu}{\tau_\nu}\right) = O\left(\frac{1}{\tau_\nu \log\tau_\nu}\right).$$

If we take $m > n \geqslant 2$, the error term in (5) is

$$\begin{aligned}O\left(\frac{1}{\tau_\nu \log^2\tau_\nu}\right) &= O\left(\frac{1}{\tau_\nu\log\tau_\nu}\frac{1}{\log m}\right) \\ &= O\left(\frac{1}{\log\tau_\nu}\frac{1}{m\log\tau_\nu}\right),\end{aligned}$$

since $\tau_\nu \geqslant \tau_{\nu_1} \geqslant m$. Also $\tau_\nu \geqslant \tau_{M+1}$, so that, by choosing M large enough, we can make the error term in (5) less than

$$\frac{1}{2m\log\tau_\nu} < \frac{\log m - \log n}{2\log\tau_\nu}$$

and also less than $$\frac{\log 2}{2\log m} \leqslant \frac{\log n}{2\log m}.$$

Thus we shall have

$$\begin{aligned}\frac{\log m - \log n}{2\log\tau_N} < \psi'(\nu) &< 1 - \frac{\log n}{\log m} + \frac{\log n}{2\log m} \\ &\leqslant 1 - \frac{\log 2}{2\log m}.\end{aligned}$$

The function $\psi'(\nu)$ is monotonic and applying Lemma 1 we have, in virtue of the above inequalities,

$$\begin{aligned}\sum\cos\{2\pi\psi(\nu)\} &= \int\cos\{2\pi\psi(x)\}\,dx + O(\log m) \\ &= O\left\{\frac{\log\tau_N}{\log(m/n)}\right\} + O(\log m).\end{aligned}$$

The contributions of these sums to $\sum_1$ are therefore

$$O\left\{\log\tau_N \sum_{n<m\leqslant\tau_N}\sum \frac{d(m)d(n)}{\sqrt{(mn)}\log(m/n)}\right\} + O\left\{\sum_{n<m\leqslant\tau_N}\sum\frac{\log m\, d(m)d(n)}{\sqrt{(mn)}}\right\}.$$

The first of these is, by Lemma 3,

$$O(\tau_N \log^4 \tau_N) = O(N \log^3 N),$$

and the second is

$$O\left\{\log \tau_N \sum_{n<\tau_N} \frac{d(n)}{\sqrt{n}} \sum_{m\leqslant\tau_N} \frac{d(m)}{\sqrt{m}}\right\} = O(\tau_N \log^3 \tau_N) = O(N \log^2 N),$$

by Lemma 2.

In the case $n = 1$ we have

$$\psi'(\nu) = \frac{\log m}{2\vartheta'(t_\nu)} < A\frac{\log m}{\log \tau_N} < A,$$

$$\psi''(\nu) = -\tfrac{1}{2}\pi \log m \frac{\vartheta''(t_\nu)}{\{\vartheta'(t_\nu)\}^3} < -A\frac{\log m}{\tau_\nu \log^3 \tau_\nu} < -\frac{A \log m}{\tau_N \log^3 \tau_N}.$$

Hence, by Theorem 1 of (I),

$$\sum \cos\{2\pi\psi(\nu)\} = O\left[\left\{\frac{\tau_N \log^3 \tau_N}{\log m}\right\}^{\frac{1}{2}}\right],$$

and the contribution of this sum to $\sum_1$ is

$$O\left\{\tau_N^{\frac{1}{2}} \log^{\frac{3}{2}} \tau_N \sum_{1<m\leqslant\tau_N} \frac{d(m)}{\sqrt{(m \log m)}}\right\} = O(\tau_N \log^2 \tau_N) = O(N \log N),$$

by Lemma 2.

Similar results hold when $m < n$, and we have altogether

$$\sum\nolimits_1 = \frac{3}{4\pi^4} N \log^4 N + O(N \log^3 N \operatorname{loglog} N).$$

4. In $\sum_2$ the ν-sum is of the form $\sum \cos\{2\pi\chi(\nu)\}$,

where
$$\chi(\nu) = \frac{1}{2\pi}(t_\nu \log m + t_{\nu+1} \log n).$$

Hence

$$\chi'(\nu) = \frac{\log m}{2\vartheta'(t_\nu)} + \frac{\log n}{2\vartheta'(t_{\nu+1})} = \frac{\log m + \log n}{\log \tau_\nu} + O\left\{\frac{\log mn}{\tau_\nu \log^3 \tau_\nu}\right\}, \tag{6}$$

as in the case of $\psi'(\nu)$.

We suppose $m > n \geqslant 2$ and consider first the case in which $mn > \tau_N \log \tau_N$. We then have

$$\tau_\nu > A(\tau_\nu \tau_{\nu+1})^{\frac{1}{2}} > A(mn)^{\frac{1}{2}} > A(\tau_N \log \tau_N)^{\frac{1}{2}}.$$

M can therefore be chosen so large that the error term in (6) is

$$O\left(\frac{1}{\tau_\nu \log^2 \tau_\nu}\right) < \frac{\operatorname{loglog} \tau_N}{2 \log \tau_N}$$

and is also less than $\dfrac{1}{2m \log m}$.

We shall then have

$$1+\frac{\log n}{\log m}+\frac{1}{2m\log m} > \chi'(\nu) > 1+\frac{\log\log \tau_N}{2\log \tau_N},$$

i.e.

$$1-\frac{\log\log \tau_N}{2\log \tau_N} > 2-\chi'(\nu) > \frac{\log m-\log n}{\log m}-\frac{1}{2m\log m}$$

$$> \frac{\log(m/n)}{2\log m}$$

$$> 0.$$

We can therefore apply Lemma 1 to the function $2x-\chi(x)$ and obtain

$$\sum \cos\{2\pi\chi(\nu)\} = \int \cos[2\pi\{2x-\chi(x)\}]\,dx + O\left(\frac{\log \tau_N}{\log\log \tau_N}\right)$$

$$= O\left\{\frac{\log m}{\log(m/n)}\right\}+O(\log \tau_N).$$

The contributions of these sums to $\sum_2$ are therefore, as in § 3.2,

$$O(N\log^3 N).$$

For the ν-sums in which $mn \leqslant \tau_N \log \tau_N$, we have

$$\chi'(\nu) = O(1),$$

$$\chi''(\nu) = -\tfrac{1}{2}\pi \log m \frac{\vartheta''(t_\nu)}{\{\vartheta'(t_\nu)\}^3}-\tfrac{1}{2}\pi\log n \frac{\vartheta''(t_{\nu+1})}{\{\vartheta'(t_{\nu+1})\}^3}$$

$$< -\frac{A\log m}{\tau_\nu \log^3 \tau_\nu}$$

$$< -\frac{A\log n}{\tau_N \log^3 \tau_N}.$$

Hence, by Theorem 1 of (I),

$$\sum \cos\{2\pi\,\chi(\nu)\} = O\left\{\left(\frac{\tau_N\log^3\tau_N}{\log n}\right)^{\frac{1}{2}}\right\},$$

and the contributions of these sums to $\sum_2$ are

$$O\left\{\tau_N^{\frac{1}{2}}\log^{\frac{3}{2}}\tau_N \sum_{n<\tau_N}\frac{d(n)}{\sqrt{(n\log n)}}\sum_{m\leqslant(\tau_N\log\tau_N)/n}\frac{d(m)}{\sqrt{m}}\right\}$$

$$= O\left\{\tau_N^{\frac{1}{2}}\log^{\frac{3}{2}}\tau_N\,\tau_N^{\frac{1}{2}}\log^{\frac{1}{2}}\tau_N\log\tau_N\sum_{n<\tau_N}\frac{d(n)}{n\sqrt{(\log n)}}\right\}$$

$$= O(\tau_N\log^3\tau_N\log^{\frac{3}{2}}\tau_N) = O(N\log^{\frac{9}{2}}N).$$

A similar argument holds for the case $n > m \geqslant 2$. If $m = 1$, $n \neq 1$, or $n = 1$, $m \neq 1$, we have a ν-sum of a form which has already been dealt with under $\sum_1$. If $m = n \geqslant 2$, we have

$$\sum \cos\{2\pi\,\chi(\nu)\} = O\left\{\left(\frac{\tau_N \log^3\tau_N}{\log m}\right)^{\frac{1}{2}}\right\},$$

and the contributions to $\sum_2$ are

$$O\left\{\tau_N^{\frac{1}{2}} \log^{\frac{3}{2}}\tau_N \sum_{2 \leqslant m \leqslant \tau_N} \frac{\{d(m)\}^2}{m\sqrt{(\log m)}}\right\} = O(\tau_N^{\frac{1}{2}} \log^{\frac{3}{2}}\tau_N \log^{\frac{7}{2}}\tau_N)$$
$$= O(\tau_N^{\frac{1}{2}} \log^5\tau_N) = O(N).$$

Finally, if $m = n = 1$, $\psi(\nu) = 0$, and this part of $\sum_2$ is $O(N)$.

Altogether we have

$$\sum_{\nu=M+1}^{N} h(t_\nu)h(t_{\nu+1}) = \frac{3}{8\pi^4} N \log^4 N + O(N \log^{\frac{7}{2}} N).$$

5. To complete the proof we note that

$$\sum_{\nu=M+1}^{N} \{f(t_\nu)\}^2 = \sum_{\nu=M+1}^{N} h(t_\nu) + O(N \log \tau_N)$$
$$= \sum_{m \leqslant \tau_N} \frac{d(m)}{\sqrt{m}} \sum_{\nu=\nu_1}^{N} \cos(t_\nu \log m) + O(N \log N)$$
$$= O(N \log^2 N),$$

since the sum is the same as that part of $\sum_1$ for which $n = 1$.

Thus we have

$$\frac{3}{8\pi^4} N \log^4 N + O(N \log^{\frac{7}{2}} N) = \sum_{\nu=M+1}^{N} h(t_\nu)h(t_{\nu+1})$$
$$= \sum [\{f(t_\nu)\}^2 + O(\log t_\nu)][\{f(t_{\nu+1})\}^2 + O(\log t_{\nu+1})]$$
$$= \sum \{f(t_\nu)f(t_{\nu+1})\}^2 + O[\log N \sum \{f(t_\nu)\}^2] + O(N \log^2 N).$$

The second term on the right is, as we have just seen, $O(N \log^3 N)$ and we can write the result as

$$\sum \{f(t_\nu)f(t_{\nu+1})\}^2 = \frac{3}{8\pi^4} N \log^4 N + O(N \log^{\frac{7}{2}} N).$$

THE REPRESENTATION OF A LARGE NUMBER AS A SUM OF 'ALMOST EQUAL' CUBES

By S. CHOWLA (*Waltair, India*)

[Received 5 November 1934]

1. WRIGHT* has shown that, if $\lambda_1,\ldots,\lambda_s$ are positive numbers whose sum is 1 and if $k \geqslant 3$, $s \geqslant (k-2)2^{k-1}+5$, then every sufficiently large n can be expressed as a sum $n = m_1^k+\ldots+m_s^k$ of s positive kth powers such that

$$\lambda_i n - m_i^k = O(n^{1-\beta}) \qquad (i = 1,2,\ldots,s)$$

with $0 < \beta < \alpha$ where α is given as a certain function of k and s; for example, if $k = 3$ and $s = 9$, then $\alpha = \frac{1}{51}$.

In this paper we prove the following special results in the same direction.

THEOREM 1. *Every sufficiently large n can be expressed in the form*

$$n = \sum_{i=1}^{8} m_i^3,$$

where $\quad \frac{1}{8}n - m_i^3 = O(n^{1-\beta}) \quad (i = 1,2,\ldots,8), \qquad 0 < \beta < \alpha,$

and (i) $\alpha = \frac{1}{4506}$ (*without any hypothesis*);

(ii) $\alpha = \frac{1}{42}$, *if the 'extended Riemann hypothesis' is true*;†

(iii) $\alpha = \frac{1}{24}$, *if hypothesis* (P) *is true*.‡

THEOREM 2. *Every sufficiently large n can be expressed in the form*

$$n = m_1^3+m_2^3+\ldots+m_9^3,$$

where $\quad \frac{1}{9}n - m_i^3 = O(n^{1-\beta}) \quad (i = 1,2,\ldots,9), \qquad 0 < \beta < \alpha,$

and (i) $\alpha = \frac{1}{42}$, *if the e. R. h. is true*;

(ii) $\alpha = \frac{1}{24}$, *if hypothesis* (P) *is true*.

Theorem 2 sharpens a special case of Wright's result (his α is $\frac{1}{51}$), assuming, however, unproved results. Theorem 1 is new.

* *Math. Zeits.* 38 (1934), 730–46.

† 'e. R. h.' = 'extended Riemann hypothesis' in what follows.

‡ Let $(a, b) = 1$. Hypothesis (P) is that for any given positive ϵ the number of primes congruent to b (to modulus a) between x and $x+x^{\epsilon}$ tends to infinity with x.

2. In what follows t is a prime such that $t \equiv 17 \pmod{48}$ and p a prime such that $p \equiv 2 \pmod 3$. We denote by $A(m)$ the assumption* that the number of primes congruent to 2 (to modulus 3) between x and $x+x^{1-1/m+\delta}$ tends to infinity with x. We shall assume that $A(m)$ is true for some fixed m.

We choose t to be the prime such that $t \equiv 17 \pmod{48}$ which is nearest to $N^{1/3(3m+1)}$. Then, since $A(m)$ is true, we can find, for sufficiently large N, at least 10 primes congruent to 2 (to modulus 3), and such that

$$(8t^3+3t^2+3t+2)p^9 < N < (8t^3+4t^2+3t+1)p^9. \tag{1}$$

Since $p^9 < N$ it follows that† at least one of the ten primes p is prime to N (large). For our further argument we therefore suppose that $(p,N) = 1$.

It follows from (1) that

$$p \sim \frac{N^{m/3(3m+1)}}{\sqrt[3]{2}}, \qquad \text{since } t \sim N^{1/3(3m+1)}. \tag{2}$$

Now $p \equiv 2 \pmod 3$, $(p,N) = 1$; hence we can‡ find β such that

$$N-\beta^3 = p^3M, \tag{3}$$

where, further,
$$tp^3 < \beta < (t+1)p^3. \tag{4}$$

From (1), (3), (4),

$$(7t^3+1)p^6 < M < (7t^3+4t^2+3t+1)p^6. \tag{5}$$

We set
$$M = 6t^3p^6+M_1. \tag{6}$$

Then, from (5),

$$(t^3+1)p^6 < M_1 < (t^3+4t^2+3t+1)p^6. \tag{7}$$

Now, we can§ find γ such that

$$M_1-\gamma^3 = 6M_2, \tag{8}$$

where
$$0 \leqslant \gamma < 96, \qquad M_2 \not\equiv 0, 7, 12, 15 \pmod{16}. \tag{9}$$

Since t is a prime and $t \equiv 2 \pmod 3$ we can|| also find γ' such that

$$M_1-\gamma'^3 \equiv O(t), \qquad 0 \leqslant \gamma' < t. \tag{10}$$

Since, further, $t \equiv 1 \pmod{16}$, it follows from (8), (9), (10) that we can find δ such that

$$M_1-\delta^3 = 6tM_3, \tag{11}$$

where
$$tp^2-96t \leqslant \delta < tp^2 \tag{12}$$

and
$$M_3 \not\equiv 0, 7, 12, 15 \pmod{16}. \tag{13}$$

* For every positive δ; naturally, $m > 1$.

† See Landau, *Primzahlen*, 555–9, for similar arguments.

‡ See ibid.; *Math. Annalen*, 66 (1909), 102–5.

§ See Landau, *Primzahlen*, 556–7.

|| As in the proof of (3).

From (7), (11), (12) it follows that

$$\frac{1}{6t}p^6 < M_3 < \frac{4t^2+3t+1}{6t}p^6+O(p^4t^2). \tag{14}$$

From (13) we have

$$M_3 = y_1^2+y_2^2+y_3^2, \tag{15}$$

where, from (2) and (14), since $m > 1$,

$$y_1, y_2, y_3 = O(\sqrt{t}p^3). \tag{16}$$

From (3), (6), (11), (15) we obtain

$$N = \beta^3+6t^3p^9+(p\delta)^3+6tp^3(y_1^2+y_2^2+y_3^2) \tag{17}$$

$$= \beta^3+(p\delta)^3+\sum_{s=1}^{3}\{(tp^3+y_s)^3+(tp^3-y_s)^3\} \tag{18}$$

$$= x_1^3+x_2^3+...+x_8^3. \tag{19}$$

From (4), (12), and (16) it now follows that

$$x_1 = tp^3+O(p^3), \qquad x_2 = tp^3+O(pt), \tag{20}$$

$$x_s = tp^3+O(\sqrt{t}p^3) \quad (3 \leqslant s \leqslant 8). \tag{21}$$

Since, for large N, $\sqrt{t}p^3 > p^3 > tp$ $(m > 1)$, it follows that, for $1 \leqslant s \leqslant 8$,

$$x_s^3-t^3p^9 = O(t^{\frac{5}{2}}p^9) \tag{22}$$

$$= O(N^{1-1/(18m+6)}), \tag{23}$$

from (2).

But

$$N-8t^3p^9 = O(t^2p^9). \tag{24}$$

From (22), (23), (24), for $1 \leqslant s \leqslant 8$,

$$x_s^3-\tfrac{1}{8}N = O(N^{1-1/(18m+6)}). \tag{25}$$

Theorem 1 is now an immediate consequence of (19) and (25) when we observe that, for arbitrary positive ϵ,

(i) $A(m)$ is true* for $m = 250+\epsilon$;

(ii) $A(m)$ is true† for $m = 2+\epsilon$, if the e. R. h. is true;

(iii) $A(m)$ is true for $m = 1+\epsilon$, if hypothesis (P) is true.

The proof of Theorem 2 is exactly similar. We start with

$$(9t^3+3t^2+3t+2)p^9 < N < (9t^3+4t^2+3t+1)p^9 \tag{26}$$

instead of (1), and obtain

$$N = (tp^3)^3+\beta^3+(p\delta)^3+\sum_{s=1}^{3}\{(tp^3+y_s)^3+(tp^3-y_s)^3\} \tag{27}$$

in place of (18).

Theorem 2 is also easily shown to be a consequence of Theorem 1.

* Heilbronn, *Math. Zeits.* 36 (1933), 394–423.

† This is well known: see, for example, Landau's *Primzahlen.*

ON THE SPHERICALLY SYMMETRIC FIELD IN RELATIVITY (III)

By B. HOFFMANN (*Rochester, N.Y.*)

[Received 28 December 1934]

1. Introduction

In two previous papers† under this title I have shown that according to the theory of relativity, and various unified field theories that fall within the general scheme of projective relativity, the most general spherically symmetric field of gravitation and electromagnetism must be static outside matter.

Recently Born and Infeld‡ have suggested a modification of the Maxwell field that leads to a finite electrostatic energy for the spherically symmetric field. A suggestion is also made§ for the inclusion of gravitational field equations, a suggestion which I have discussed elsewhere.||

In the present paper I show that, if we adopt the field equations suggested by Born and Infeld for the complete gravitational and electromagnetic field, the spherically symmetric field must still be static. The most general spherically symmetric fields for two possible sets of field equations are then determined.

I shall adhere as far as possible to the notation of the previous papers of this series, and shall number equations in conformity with previous numbering as far as the early parts of § 5 of this paper, placing a double or triple accent on the number of each equation that has an analogue in either of the other papers.

2. The field equations

If we make the necessary alterations in Born and Infeld's notation to bring it into conformity with that of the present series, and if we also include a cosmological term, we may write the basic field equations,†† with a suitable choice of units, as

$$G_{ab}+E_{ab}+\lambda g_{ab} = 0, \qquad \Phi(ab)$$

$$\frac{\partial}{\partial x^b}\{\sqrt{(-g)}(F^{ab}-GF^{*ab})/\sqrt{(1+F-G^2)}\} = 0, \qquad \Phi''(a)$$

† B. Hoffmann, *Quart. J. of Math.* (Oxford), 3 (1932), 226–37, and 4 (1933), 179–83, referred to respectively as (I) and (II).

‡ *Proc. Royal Soc.* 144 A (1934), 425, referred to as B.I. § B.I. 435.

|| *Proc. Royal Soc.* 148 A (1935), 353. †† Cf. B.I., (3.1), (3.4), (3.6), and (4.5).

where λ is a constant, $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$,

R_{ab} is the Ricci tensor of the line element $ds^2 = g_{ab}\,dx^a dx^b$,

$$E_{ab} = g_{ab}\{1-\surd(1+F-G^2)-G^2/\surd(1+F-G^2)\}+ \\ +g^{cd}F_{ac}F_{bd}/\surd(1+F-G^2), \tag{1''.1}$$

$$F = \tfrac{1}{2}F_{ab}F^{ab}, \tag{1''.2}$$

$$G = (F_{23}F_{14}+F_{31}F_{24}+F_{12}F_{34})/\surd(-g), \tag{1''.3}$$

$$F^{*ab} = \epsilon^{abcd}F_{cd}/2\surd(-g), \tag{1''.4}$$

$$F_{ab} = \frac{\partial\phi_a}{\partial x^b} - \frac{\partial\phi_b}{\partial x^a}, \tag{2}$$

$$g = |g_{ab}|.$$

The equations (2) may be replaced by the equivalent equations

$$\frac{\partial F_{bc}}{\partial x^a} + \frac{\partial F_{ca}}{\partial x^b} + \frac{\partial F_{ab}}{\partial x^c} = 0. \tag{$\Phi(abc)$}$$

3. Spherical symmetry

The argument of § 3 of (I) is valid for the present case so that we shall assume the general spherically symmetric line element

$$ds^2 = A\,dt^2 - B\,dr^2 - C(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{3}$$

where A, B, C are functions of r and t alone, and shall place no other restriction on the field beyond whatever is implied by the field equations.

The field equations $\Phi(ab)$ are of the general form

$$G_{ab} + g_{ab}\Omega + K_{ab} = 0, \tag{$\Phi_1(ab)$}$$

where Ω is a scalar, and

$$K_{ab} \equiv g^{cd}F_{ac}F_{bd}/\surd(1+F-G^2).$$

Since the g_{ab} of (3) are spherically symmetric, the G_{ab} obtained from them will satisfy the conditions of spherical symmetry given in equations (8) of (I). Thus we must have

$$\left.\begin{array}{c} G_{12}, G_{13}, G_{23}, G_{24}, G_{34} = 0 \\ G_{33} = G_{22}\sin^2\theta \\ G_{11}, G_{22}, G_{44}, G_{14} \text{ independent of } \theta \text{ and } \phi \end{array}\right\}; \tag{8}$$

and we shall have proved the line element to be essentially static, if we can further show that†

$$G_1^1 = G_4^4, \qquad G_{14} = 0.$$

† Cf. (I) 231.

But the g_{ab} satisfy the conditions

$$g_{ab} = 0 \quad (a \neq b), \qquad g_1^1 = g_4^4,$$

so that, from $\Phi_1(ab)$, it will ensure that the gravitational part of the field is essentially static, if we can prove that the K_{ab} must satisfy the relations

$$K_1^1 = K_4^4, \qquad K_{14} = 0.$$

4. Proof that the gravitational part of the field is static

For the line element and coordinate-system of (3) we have

$$\left.\begin{aligned}
\sqrt{(1+F-G^2)}K_{11} &= -(F_{12})^2/C-(F_{13})^2/C\sin^2\theta+(F_{14})^2/A,\\
\sqrt{(1+F-G^2)}K_{22} &= -(F_{12})^2/B-(F_{23})^2/C\sin^2\theta+(F_{24})^2/A,\\
\sqrt{(1+F-G^2)}K_{33} &= -(F_{13})^2/B-(F_{23})^2/C+(F_{34})^2/A,\\
\sqrt{(1+F-G^2)}K_{44} &= -(F_{14})^2/B-(F_{24})^2/C-(F_{34})^2/C\sin^2\theta,\\
\sqrt{(1+F-G^2)}K_{12} &= -F_{13}F_{23}/C\sin^2\theta+F_{14}F_{24}/A,\\
\sqrt{(1+F-G^2)}K_{13} &= F_{12}F_{23}/C+F_{14}F_{34}/A,\\
\sqrt{(1+F-G^2)}K_{14} &= F_{12}F_{24}/C+F_{13}F_{34}/C\sin^2\theta,\\
\sqrt{(1+F-G^2)}K_{23} &= -F_{12}F_{13}/B+F_{24}F_{34}/A,\\
\sqrt{(1+F-G^2)}K_{24} &= -F_{12}F_{14}/B+F_{23}F_{34}/C\sin^2\theta,\\
\sqrt{(1+F-G^2)}K_{34} &= -F_{13}F_{14}/B+F_{23}F_{24}/C.
\end{aligned}\right\} \quad (9'')$$

From these values of the components of K_{ab} we see that, if we assume that the radical remains finite on the ground that we are not interested in infinite fields, the five field equations $\Phi_1(12)$, $\Phi_1(13)$, $\Phi_1(23)$, $\Phi_1(24)$, $\Phi_1(34)$ reduce to their counterparts in (I). We may therefore follow the argument of (I) that is based on these five equations.* And this argument now shows that these five field equations, together with equations (8) and (9″), place conditions on the components of F_{ab} that lead to the two possible cases

$$\text{(i)}\ \ F_{12}, F_{34}, F_{13}, F_{24} = 0;$$

$$\text{(ii)}\ \ F_{14}, F_{23} = 0.$$

We consider these two cases in turn:

(i) Here $F_{14} = -F_{41}$ and $F_{23} = -F_{32}$ are the only non-zero components of F_{ab}; we thus have, from (9″),

$$K_{14} = 0, \qquad K_{11} = (F_{14})^2/A\sqrt{(1+F-G^2)},$$

$$K_{44} = -(F_{14})^2/B\sqrt{(1+F-G^2)},$$

* Cf. (I) 230–1.

so that the conditions

$$K_{14} = 0, \qquad K_1^1 = K_4^4$$

are satisfied in this case, and the line element is therefore essentially static.

(ii) Here F_{14} and F_{23} vanish.

Now
$$g_{33} = g_{22}\sin^2\theta$$

so that, from $\Phi_1(ab)$ and (8), we may obtain

$$K_{33} = K_{22}\sin^2\theta$$

which, from (9″), can, in the present case, be written as

$$(F_{12})^2/B+(F_{34})^2/A\sin^2\theta = (F_{24})^2/A+(F_{13})^2/B\sin^2\theta, \tag{12}$$

and this is the same as equation (12) of (I).

We may therefore follow the argument of case (ii) of (I) as far as the result* that

$$A(F_{13})^2 = B(F_{34})^2, \tag{13}$$

$$A(F_{12})^2 = B(F_{24})^2, \tag{14}$$

where, as in (I), we must be careful to take the same signs in the square roots of these equations.

For the present case, if we make use of (13) and (14), we find that

$$\begin{aligned} F &= (F_{12})^2/BC+(F_{13})^2/BC\sin^2\theta-(F_{24})^2/AC-(F_{34})^2/AC\sin^2\theta \\ &= (F_{12})^2/BC+(F_{13})^2/BC\sin^2\theta-(F_{12})^2/BC-(F_{13})^2/BC\sin^2\theta = 0, \end{aligned}$$

and also that

$$\begin{aligned} G &= (F_{31}F_{24}+F_{12}F_{34})/\sqrt{(-g)} \\ &= \{-F_{13}F_{12}\sqrt{(A/B)}+F_{12}F_{13}\sqrt{(A/B)}\}/\sqrt{(-g)} = 0. \end{aligned}$$

Thus, from (9″), we now find that

$$K_{11} = -(F_{12})^2/C-(F_{13})^2/C\sin^2\theta;$$

and from the conditions similar to (8) that must be satisfied by K_{ab} it follows that this quantity must be independent of θ and ϕ.

But the remainder of the argument of (I) for case (ii) is based on the fact† that just this quantity K_{11} is independent of θ and ϕ, and upon the field equations $\Phi(a)$ of (I) and the equations (2), that may be written in the equivalent form $\Phi(abc)$; and since it has been established that F and G are zero for the present case, so that the field equations $\Phi''(a)$ reduce exactly to the field equations $\Phi(a)$ of (I), while the equations (2) of this paper are identical with their counterpart in (I), it follows that we may pursue for the present case the argument

* Ibid. 232. † Cf. (I) 232, lower portion.

of the case (ii) of (I) as far as the conclusion† that all the components of F_{ab} must vanish. The line element will therefore be essentially static for the present case.

Thus we have shown that in all cases the line element must be essentially static, and that the only possible non-zero components of F_{ab} are $F_{14} = -F_{41}$ and $F_{23} = -F_{32}$.

5. The most general spherically symmetric field

It is now a simple matter to obtain the most general spherically symmetric field.

We may follow the argument of § 5 of (I) to a certain extent. Since the line element must be essentially static, we can find a transformation of coordinates of the type

$$r \to r(R,T), \qquad \theta \to \theta, \qquad \phi \to \phi, \qquad t \to t(R,T) \tag{16}$$

that will lead to a line element of the form

$$ds^2 = \mathfrak{A}\,dT^2 - \mathfrak{B}\,dR^2 - R^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are functions of R alone. As in (I) we shall change the notation so that the new line element may be written as

$$ds^2 = A\,dt^2 - B\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{17}$$

where A and B are functions of r alone, and A, B, r, t are, in general, different from the A, B, r, t of the previous sections. Since the transformation of coordinates (16) cannot alter the form of the conditions of spherical symmetry,‡ we may repeat the whole argument of the previous sections for the new coordinate-system and obtain the result that the only non-zero components of the new F_{ab} are $F_{14} = -F_{41}$ and $F_{23} = -F_{32}$.

For the present case, the equations $\Phi(abc)$ reduce respectively to

$$\frac{\partial F_{23}}{\partial t} = 0, \qquad \frac{\partial F_{41}}{\partial \phi} = 0, \qquad \frac{\partial F_{41}}{\partial \theta} = 0, \qquad \frac{\partial F_{23}}{\partial r} = 0,$$

whence we find that

$$F_{23} = F_{23}(\theta, \phi), \qquad F_{14} = F_{14}(r,t). \tag{18}$$

Again, the field equations $\Phi''(a)$ become respectively

$$\frac{\partial}{\partial t}\{\sqrt{(-g)}(F^{14} - GF^{*14})/\sqrt{(1+F-G^2)}\} = 0,$$

$$\frac{\partial}{\partial \phi}\{\sqrt{(-g)}(F^{23} - GF^{*23})/\sqrt{(1+F-G^2)}\} = 0,$$

† Cf. (I) top of p. 234. ‡ Cf. (I) 227, below equation (4).

$$\frac{\partial}{\partial\theta}\{\sqrt{(-g)}(F^{32}-GF^{*32})/\sqrt{(1+F-G^2)}\} = 0,$$

$$\frac{\partial}{\partial r}\{\sqrt{(-g)}(F^{41}-GF^{*41})/\sqrt{(1+F-G^2)}\} = 0.$$

But we now have

$$F = -(F_{14})^2/AB+(F_{23})^2/r^4\sin^2\theta,$$

$$G = F_{14}F_{23}/\sqrt{(AB)}r^2\sin\theta,$$

so that

$$1+F-G^2 = \{AB-(F_{14})^2\}\{r^4\sin^2\theta+(F_{23})^2\}/ABr^4\sin^2\theta.$$

Thus, since

$$F^{*14} = F_{23}/\sqrt{(-g)}, \qquad F^{*23} = F_{14}/\sqrt{(-g)},$$

the $\Phi''(a)$ equations reduce to

$$\frac{\partial}{\partial t}[F_{14}\sqrt{\{r^4\sin^2\theta+(F_{23})^2\}}/\sqrt{\{AB-(F_{14})^2\}}] = 0, \qquad \Phi''(1)$$

$$\frac{\partial}{\partial\phi}[F_{23}\sqrt{\{AB-(F_{14})^2\}}/\sqrt{\{r^4\sin^2\theta+(F_{23})^2\}}] = 0, \qquad \Phi''(2)$$

$$\frac{\partial}{\partial\theta}[F_{23}\sqrt{\{AB-(F_{14})^2\}}/\sqrt{\{r^4\sin^2\theta+(F_{23})^2\}}] = 0, \qquad \Phi''(3)$$

$$\frac{\partial}{\partial r}[F_{14}\sqrt{\{r^4\sin^2\theta+(F_{23})^2\}}/\sqrt{\{AB-(F_{14})^2\}}] = 0. \qquad \Phi''(4)$$

From $\Phi''(1)$, since, by (18), F_{23} is independent of t, we deduce that

$$F_{14}/\sqrt{\{AB-(F_{14})^2\}} \text{ is independent of } t.$$

But A and B are functions of r alone, and, by (18), F_{14} is at most a function of r and t, so that the above quantity must be a function of r alone. If we denote this unknown function by $\Omega(r)$, we have

$$(F_{14})^2 = AB\Omega^2/(1+\Omega^2),$$

which shows that

$$F_{14} = F_{14}(r). \qquad (19)$$

Again, from $\Phi''(2)$ and (18), we find in the same way that

$$F_{23} = F_{23}(\theta). \qquad (20)$$

From $\Phi''(3)$ we must have

$$F_{23}\sqrt{\{AB-(F_{14})^2\}}/\sqrt{\{r^4\sin^2\theta+(F_{23})^2\}} \text{ independent of } \theta;$$

but since A, B are functions of r alone, and, by (19) and (20), F_{23}, F_{14} are respectively functions of θ and r alone, this implies that

$$F_{23}/\sqrt{\{r^4\sin^2\theta+(F_{23})^2\}} = R(r).$$

Solving for F_{23} we find

$$F_{23} = Rr^2 \sin\theta/\sqrt{(1-R^2)}.$$

But we have already seen that F_{23} is a function of θ alone, so that it cannot involve r. Therefore we must have merely

$$F_{23} = \mu \sin\theta, \tag{21}$$

where μ is a constant.

Finally, from $\Phi''(4)$, (19), and (20), we obtain in a similar manner the fact that

$$F_{14}\sqrt{\{r^4 \sin^2\theta+(F_{23})^2\}}/\sqrt{\{AB-(F_{14})^2\}} = \Theta(\theta),$$

and if we make use of the value (21) that we have just obtained for F_{23}, we find that

$$F_{14}\sqrt{(r^4+\mu^2)}/\sqrt{\{AB-(F_{14})^2\}} = \Theta(\theta)\operatorname{cosec}\theta.$$

But the left-hand side is evidently independent of θ. Hence the right-hand side must reduce to a constant, say ϵ, and now we obtain

$$F_{14} = \epsilon\sqrt{(AB)}/\sqrt{(r^4+\epsilon^2+\mu^2)}. \tag{22}$$

Thus from the two sets of equations $\Phi(abc)$ and $\Phi''(a)$ we have obtained integrals expressible in the two equations (21) and (22). We must now apply these results to the non-trivial equations of the set $\Phi(ab)$.

With the values for F_{14} and F_{23} given in (21) and (22), we have

$$\sqrt{(1+F-G^2)} = (r^4+\mu^2)/r^2\sqrt{(r^4+\epsilon^2+\mu^2)}, \tag{23}$$

$$G^2 = \epsilon^2\mu^2/r^4(r^4+\epsilon^2+\mu^2), \tag{24}$$

and, since now

$$g^{cd}F_{1c}F_{1d} = B\epsilon^2/(r^4+\epsilon^2+\mu^2),$$
$$g^{cd}F_{2c}F_{2d} = -\mu^2/r^2,$$
$$g^{cd}F_{4c}F_{4d} = -A\epsilon^2/(r^4+\epsilon^2+\mu^2),$$

it is not difficult to show that

$$E_{11} = -B\{1-\sqrt{(r^4+\mu^2+\epsilon^2)}/r^2\}, \tag{25 α}$$

$$E_{22} = -r^2\{1-r^2/\sqrt{(r^4+\mu^2+\epsilon^2)}\}, \tag{26 α}$$

$$E_{44} = A\{1-\sqrt{(r^4+\mu^2+\epsilon^2)}/r^2\}. \tag{27 α}$$

If we now write e^ν for A and e^κ for B we find* that

$$R_{11}-\tfrac{1}{2}g_{11}R+\lambda g_{11} = -\nu'/r-1/r^2+e^\kappa(1/r^2-\lambda), \tag{25 β}$$

* We use here the values of the components of R_{ab} given by Eddington, *Mathematical Theory of Relativity*, 2nd ed., p. 85 (Eddington's $G_{\mu\nu}$ is our R_{ab}). I have used κ instead of Eddington's λ to avoid confusion with the cosmological constant.

$$R_{22}-\tfrac{1}{2}g_{22}R+\lambda g_{22}$$
$$= e^{-\kappa}\{-\tfrac{1}{2}r^2\nu''+\tfrac{1}{4}r^2\kappa'\nu'-\tfrac{1}{4}r^2\nu'^2+\tfrac{1}{2}r(\kappa'-\nu')\}-r^2\lambda, \tag{26β}$$
$$R_{44}-\tfrac{1}{2}g_{44}R+\lambda g_{44} = e^{\nu-\kappa}\{-\kappa'/r+1/r^2-e^{\kappa}(1/r^2-\lambda)\}. \tag{27β}$$

If we make use of the values we have just computed for components of E_{ab} and of $(R_{ab}-\frac{1}{2}g_{ab}R+\lambda g_{ab})$, we find at once that the field equations $\Phi(11)$ and $\Phi(44)$ yield
$$\kappa'+\nu' = 0, \tag{28}$$
which we may integrate without loss of generality to give
$$\kappa+\nu = 0. \tag{29}$$

If we make use of this result to eliminate κ from the remaining field equations, we are left with the two equations
$$e^{\nu}(r\nu'+1)-1+r^2(1+\lambda)-\sqrt{(r^4+\mu^2+\epsilon^2)} = 0, \qquad \Phi''(11)$$
$$e^{\nu}(\nu''+\nu'^2+2\nu'/r)+2(1+\lambda)-2r^2/\sqrt{(r^4+\mu^2+\epsilon^2)} = 0, \qquad \Phi''(22)$$
which must be simultaneously satisfied by ν.

It is easily verified that if we differentiate equation $\Phi''(11)$ with respect to r we obtain precisely equation $\Phi''(22)$ apart from a factor r. Hence any solution of $\Phi''(11)$ will automatically satisfy $\Phi''(22)$.

Now $\Phi''(11)$ may be written as
$$\frac{d}{dr}(re^{\nu}) = 1-r^2(1+\lambda)+\sqrt{(r^4+\mu^2+\epsilon^2)},$$
which at once gives the integral
$$e^{\nu} = 1-2m/r-\tfrac{1}{3}(1+\lambda)r^2+\frac{1}{r}\int\limits_0^r \sqrt{(r^4+\mu^2+\epsilon^2)}\,dr,$$
where m is a constant of integration.

Thus the most general spherically symmetric field is given by
$$\left.\begin{aligned} ds^2 &= A\,dt^2-A^{-1}\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2),\\ F_{14} &= \epsilon/\sqrt{(r^4+\mu^2+\epsilon^2)}, \qquad F_{23} = \mu\sin\theta,\\ \text{where}\quad A &= 1-2m/r-\tfrac{1}{3}(1+\lambda)r^2+\frac{1}{r}\int\limits_0^r\sqrt{(r^4+\mu^2+\epsilon^2)}\,dr. \end{aligned}\right\} \tag{30}$$

6. Interpretation of the field

From an argument similar to that of § 6 of (I), it is evident that the field we have obtained represents that which would arise from a spherically symmetric distribution of 'matter' having total mass m,

total electric-pole-strength ϵ, and total magnetic-pole-strength μ. Remarks similar to those of § 6 of (I) are applicable to the possibility of the existence of isolated magnetic poles implied by our solution.

In the case treated in (I), the field would still be static even though the matter producing it might have any motion whatever, provided only that it remained spherically symmetric. This was because the field equations considered in (I) applied only to the external field and would require to be augmented by a material energy tensor before they were applicable to the region occupied by matter.

But in the Born-Infeld theory, matter is considered as purely electromagnetic and gravitational in origin. There is no room for an additional tensor in the field equations to represent the motion and distribution of matter, since this is already represented by the gravitational and electromagnetic tensors. Thus the field equations must be valid for all regions, whether occupied by matter or not, and we must conclude from our work that a perfectly spherically symmetric distribution of matter is completely defined by three parameters m, ϵ, μ, and that it is completely static. Such a 'frozen' distribution will have the same permanence as a vortex in a non-viscous medium; and, like it, will not be exactly realized in nature.

7. The alternative Born-Infeld equations

Born and Infeld have also considered a slightly different set of field equations* based on a Lagrangian for the electromagnetic field that does not contain the G-term of the Lagrangian we have been using. The corresponding field equations will be obtained from those so far considered in this paper by replacing G by zero wherever it occurs.

The argument of §§ 1–4 may be applied to the new field equations with very little change and will lead to the same result as in the case of the previous field equations, namely, that the line element must be essentially static and that the only possible non-zero components of F_{ab} are $F_{14} = -F_{41}$ and $F_{23} = -F_{32}$.

We may now make the transformation of coordinates of § 5 and continue along the lines of that section with the new field equations. We shall obtain the results (18) from the field equations $\Phi(abc)$, and from the new field equations $\Phi'''(a)$ we shall then obtain the results that

$$F_{14} = F_{14}(r), \quad \text{from } \Phi'''(1),$$

* Cf. B.I. 432, equation (2.28).

$$F_{23} = F_{23}(\theta), \quad \text{from } \Phi'''(2),$$

$$F_{23} = \mu \sin\theta, \quad \text{from } \Phi'''(3),$$

$$\frac{\partial}{\partial r}[r^2F_{14}/\surd\{AB-(F_{14})^2+AB\mu^2/r^4\}] = 0; \qquad \Phi'''(4)$$

the last gives

$$r^2F_{14}/\surd\{AB-(F_{14})^2+AB\mu^2/r^4\} = \text{constant, } \epsilon \text{ say,}$$

leading to $$F_{14} = \epsilon\surd\{AB(1+\mu^2/r^4)\}/\surd(r^4+\epsilon^2). \qquad (22')$$

With these values of F_{14} and F_{23} we now have

$$\surd(1+F) = \surd\{(r^4+\mu^2)/(r^4+\epsilon^2)\} \qquad (23')$$

and, since now

$$g^{cd}F_{1c}F_{1d} = \epsilon^2B(1+\mu^2/r^4)/(r^4+\epsilon^2), \qquad g^{cd}F_{2c}F_{2d} = -\mu^2/r^2,$$

$$g^{cd}F_{4c}F_{4d} = -\epsilon^2A(1+\mu^2/r^4)/(r^4+\epsilon^2),$$

it is not difficult to show that for the present case we have

$$E'_{11} = -B[1-\surd\{(r^4+\mu^2)(r^4+\epsilon^2)\}/r^4], \qquad (25'\,\alpha)$$

$$E'_{22} = -r^2[1-\surd\{(r^4+\mu^2)/(r^4+\epsilon^2)\}+(\mu^2/r^4)\surd\{(r^4+\epsilon^2)/(r^4+\mu^2)\}], \quad (26'\,\alpha)$$

$$E'_{44} = A[1-\surd\{(r^4+\mu^2)(r^4+\epsilon^2)\}/r^4]. \qquad (27'\,\alpha)$$

If we use the results expressed in equations $(25\,\beta)$ and $(27\,\beta)$ together with those of $(25'\,\alpha)$ and $(27'\,\alpha)$ the field equations $\Phi(11)$ and $\Phi(44)$ yield as before

$$\kappa+\nu = 0, \qquad (29)$$

and, if we now eliminate κ from the remaining field equations, we are left with the two equations

$$e^\nu(r\nu'+1)-1+r^2(1+\lambda)-\surd\{(r^4+\mu^2)(r^4+\epsilon^2)\}/r^2 = 0, \qquad \Phi'''(11)$$

$$e^\nu(\nu''+\nu'^2+\nu'/r)+2(1+\lambda)-2\surd\{(r^4+\mu^2)/(r^4+\epsilon^2)\}+$$
$$+2\mu^2\surd\{(r^4+\epsilon^2)/(r^4+\mu^2)\}/r^4 = 0. \quad \Phi'''(22)$$

It will be found that $\Phi'''(22)$ is essentially the derivative of $\Phi'''(11)$, so that we need only solve the latter. It can be written as

$$\frac{d}{dr}(re^\nu) = 1-r^2(1+\lambda)+\surd\{(r^4+\mu^2)(r^4+\epsilon^2)\}/r^2,$$

which at once gives the integral

$$e^\nu = 1-2m/r-(1+\lambda)r^2/3+\frac{1}{r}\int\limits_0^r [\surd\{(r^4+\mu^2)(r^4+\epsilon^2)\}/r^2]\,dr.$$

Thus the most general spherically symmetric field for the alternative field equations is given by

$$\left.\begin{aligned} ds^2 &= A\,dt^2 - A^{-1}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \\ F_{14} &= \epsilon\sqrt{(r^4+\mu^2)}/r^2\sqrt{(r^4+\epsilon^2)}, \qquad F_{23} = \mu\sin\theta, \\ &\text{where} \\ A &= 1 - 2m/r - (1+\lambda)r^2/3 + \frac{1}{r}\int\limits_0^r [\sqrt{\{(r^4+\mu^2)(r^4+\epsilon^2)\}}/r^2]\,dr. \end{aligned}\right\} \quad (30')$$

8. Interpretation of the alternative field

The new field is evidently that corresponding to a spherically symmetric distribution of matter having total mass m, total electric-pole-strength ϵ, and total magnetic-pole-strength μ. All the remarks of § 6 apply to the present case.

9. Comparison of the two fields

In the general spherically symmetric field of the ordinary theory of relativity obtained in (I),* the gravitational effect of the magnetic-pole-strength μ is of exactly the same type as that of the electrical-pole-strength ϵ. In fact, the gravitational effect depends solely upon the quantity $\epsilon^2+\mu^2$. It is an interesting fact that this is also true of the field (30) of the first set of field equations of the present paper. It is not the case for the field (30′) given by the second set of field equations.

In the purely electrostatic case the G-term is easily seen to be identically zero, so that, as was pointed out by Born and Infeld,† there will be no difference in the electrostatic fields given by the two sets of field equations. For this reason Born and Infeld were unable to decide which set of field equations was to be preferred on physical grounds.

We have shown, however, that when we consider the most general spherically symmetric case the difference between the field equations shows up in the resulting fields. It will be seen that the two fields become identical when we let μ become zero; this is, of course, to be expected, since when μ is zero the two sets of equations become equivalent.

The Born-Infeld theory was proposed in order to avoid the infinities of the Maxwell theory. But, if μ differs from zero, the second

* (I) 236, equation (20). † B.I. 433, top.

set of field equations fails to avoid the infinity at $r = 0$ in the electric intensity, though there is no such infinity in the field given by the first set of field equations. Hence, if we permit values of μ other than zero, we must give preference on physical grounds to the first set of field equations. If, however, we do not admit the possibility of isolated magnetic poles—or, more accurately, of isolated magnetic-pole-strengths in conjunction with electric-pole-strengths—we cannot decide between the two sets of field equations on the basis of the work of this paper and the question of which is to be preferred remains an open one.

[*Note added in proof 13 April 1935.* The value of the electromagnetic energy-tensor E_{ab} given in equation (1''.1) is incorrect and should be replaced by its negative. Though this makes no essential difference to the argument of the present paper, and its effect on the final fields is easily determined, it is of importance in connexion with an alternative physical interpretation that I have put forward. Details are given in an article 'Gravitational and Electromagnetic Mass in the Born-Infeld Electrodynamics' which is to appear in *The Physical Review.*]